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Optimal switching under a hybrid diffusion model and applications to stock trading ☆

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ABSTRACT

This paper is concerned with the optimal switching problem under a hybrid diffusion (or, regime switching) model in an infinite horizon. The state of the system consists of a number of diffusions coupled by a finite-state continuous-time Markov chain. Based on the dynamic programming principle, the value function of our optimal switching problem is proved to be the unique viscosity solution to the associated system of variational inequalities. The optimal switching strategy, indicating when and where it is optimal to switch, is given in terms of the switching and continuation regions. In many applications, the underlying Markov chain has a large state space and exhibits two-time-scale structure. In this case, a singular perturbation approach is employed to reduce the computational complexity involved. It is shown that as the time-scale parameter ε goes to zero, the value function of the original problem converges to that of a limit problem. The limit problem is much easier to solve, and its optimal switching solution leads to an approximate solution to the original problem. Finally, as an application of our theoretical results, an example concerning the stock trading problem in a regime switching market is provided. It is emphasized that, this paper is the first time to introduce the optimal switching as a general framework to study the stock trading problem, in view of their inherent connection. Both optimal trading rule and convergence result are numerically demonstrated in this example.

1. Introduction

Optimal switching is the problem of determining an optimal sequence of stopping times at which to change the regime of the control process to another. When there are multiple (more than two) regimes, it needs to decide not only when to switch but also where to switch. Traditionally, the theory of optimal switching has been extensively studied along the line of classical variational inequalities method; see, for example, Tang and Yong (1993) and El Asri and Hamadène (2009) (finite horizon), Pham (2007), Pham, Ly Vath, and Zhou (2009) and Johnson and Zervos (2010) (infinite horizon), among others. Generally speaking, the associated system of variational inequalities is a set of ordinary (respectively, partial) differential equations of second order with inner-connected obstacles in the infinite (respectively, finite) horizon situation. The other important approach is the utilization of reflected backward stochastic differential equations (BSDEs for short). The pioneering work on this subject is Hamadène and Jeanblanc (2007), which studied a starting and stopping problem (or, two-regime switching problem). The results were then extended in Djehiche, Hamadène, and Popier (2009) to multiple-regime case. In addition, Hu and Tang (2010) considered BSDEs with oblique reflections to solve the optimal switching problem directly, without using the Snell envelope of processes as in Djehiche et al. (2009) and Hamadène and Jeanblanc (2007). In contrast with the variational inequalities method, the reflected BSDEs approach is able to deal with the finite horizon optimal switching problem in non-Markovian context. However, for the infinite horizon case, there are few results; see, for example, Hamadène, Lepeltier, and Wu (1999), which are not effective to treat the optimal switching problem.

Apart from the mathematical interest in its own right, optimal switching enjoys a wide range of applications. An immediate example is electricity production management. Typically, the cost for storing electrical power is expensive. A power plant has to

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decide when and how to generate electricity according to market demand, for example, operate at full capacity for high demand and partial capacity (even shut down the generators) for low demand. There are normally fixed costs attached with these switchings. The plant management needs to make decisions by taking market demand and switching costs into consideration; see Carmona and Ludkovski (2008) and Deng and Xia (2006) for related discussions. Similar problems for investment strategy of a sequence of entry and exit (or, buy and sell) decisions, were proposed and addressed in Brekke and Øksendal (1994), Duckworth and Zervos (2001), and Zervos, Johnson, and Alazemi (2013). It is worth noting that, though having not ever been pointed out explicitly, the stock trading problems considered in Kong, Zhang, and Yin (2011) and Zhang and Zhang (2008) are in nature optimal switching problems, if we observe that the successive "buy" and "sell" actions made by the agent in the stock market can be regarded as a sequence of switching decisions.

In this paper, we consider the optimal switching problem in an infinite horizon under a hybrid diffusion model consisting of a set of diffusions coupled by a finite-state continuous-time Markov chain. Compared with pure diffusion models, hybrid diffusion models capture more directly the dynamics of events that are less frequent (occasional) but nevertheless more significant to longer-term system behavior. Taking the stock market for example, the Markov chain can represent discrete events such as market trends and other economic factors that are difficult to be incorporated into a pure diffusion model, and allow the key parameters of the stock to respond to the general market movements; see Zhang (2001). Note that there are two kinds of "switching" in this paper: the first one is the switching decisions made by the controller, and the second one is the switching among system regimes that are modulated by the Markov chain. We shall follow the variational inequalities method to solve our optimal switching problem. (In the case of finite horizon, Tao, Wu, and Zhang (2015) used obliquely reflected BSDEs with Markov chains to study a similar problem.) Based on the dynamic programming principle, the value function of our optimal switching problem is proved to be the unique viscosity solution to the associated system of variational inequalities. Moreover, the optimal switching strategy is given in terms of the switching and continuation regions, which are characterized by the "obstacle parts" of the variational inequalities.

In many applications, the state space of the Markov chain is inevitably large. As a consequence, to solve the corresponding system of variational inequalities is difficult both theoretically and computationally. In this paper, we use a singular perturbation approach based on a two-time-scale model to reduce the complexity involved. To be precise, in a lot of practical settings, quite often different elements in a large system evolve at different rates. Some of them vary rapidly, and others change slowly. The dynamic system evolves as if different elements use different clocks or time scales. The main idea is to formulate the problem using a Markov chain with two-time-scale structure, i.e., the states of the Markov chain are divided into a number of groups so that the chain jumps rapidly within each group and slowly between the groups. Then the variables associated with the fast scale are "averaged out" and replaced by the corresponding stationary distributions. This gives rise to a limit problem identified by the stationary distributions of the fast parts and the slowly varying component. The optimal switching strategy for this limit problem provides an approximation to the original problem, in the sense that the value function of the original problem converges to that of the limit problem. Further details and more illustrations about two-time-scale systems can be found in, for example, Zhang, Zhang, and Yin (2017) and the book by Yin and Zhang (2013).

Finally, we apply the general results obtained to the stock trading problem in a regime switching market, taking advantage of its inherent connection with optimal switching problem. Typically, the stock trading problems were studied within the framework of verification theorem, by "guessing" a priori the form of the corresponding system of variational inequalities; see Kong et al. (2011) and Zhang and Zhang (2008). However, this method relies heavily on the existence of a smooth solution to the system of variational inequalities. Consequently, it fails to treat more complex situations, such as the underlying Markov chain has more than two states, where a smooth solution is difficult to obtain. In this paper, we first cast the classical stock trading problem into optimal switching setting, which makes our general results available to this special case. Then numerical scheme based on finite difference method is proceeded to compute the value function and optimal trading rule according to the variational inequalities. It turns out that, when the Markov chain has two states (the market can be regarded as a bull market or a bear market at a given time), the optimal trading rule is a trend following strategy, which validates the theoretical justification of Dai, Zhang, and Zhu (2010) and Kong et al. (2011).

On the other hand, equity investors in the market can be classified into two categories in accordance with their trading strategies: long-term investors and short-term investors. A long-term investor is concerned with the longtime behavior of a stock and makes decisions based on weekly or monthly price charts of the stock, whereas a short-term investor deals with only short-term returns based on daily or even hourly price behavior of the equity. Their time scales are in sharp contrast. In this case, the two-time-scale approach developed in this paper would be helpful for long-term investors to stay focused on the longtime outlooks of the stock market.

To recapitulate, the main contributions of this paper include the following: (i) We develop the variational inequalities method for the optimal switching problem under a hybrid diffusion model. Compared with the same problem but under a pure diffusion model (see Pham, 2007), the proofs of the dynamic programming principle and existence and uniqueness of viscosity solution to the variational inequalities system are more delicate. (ii) When the Markov chain has a two-time-scale structure, the viscosity solution approach is employed to verify the convergence of the value function. This allows us to obtain the convergence in a general setting and helps us to characterize the structure of the limit system. (iii) The optimal switching framework we introduce provides an appropriate and general model to study the stock trading problem. Such an approach establishes a natural connection of the stock trading problem and standard stochastic control theory, which offers an opportunity for treating more complicated situations.

This paper is organized as follows. In Section 2, we formulate the problem under consideration and make assumptions. In Section 3, we give the dynamic programming principle in the present case, and the value function is proved to be the unique viscosity solution to the associated system of variational inequalities. Section 4 uses a two-time-scale structure and obtain the corresponding convergence. Then, in Section 5, we provide a numerical example concerning the stock trading problem to demonstrate our theoretical results. Finally, Section 6 concludes the paper.

2. Problem formulation and preliminaries

Let $(\Omega, \mathcal{F}, P)$ be a fixed probability space on which are defined a standard one-dimensional Brownian motion $B(t)$, $t \geq 0$ and a Markov chain $\alpha(t)$, $t \geq 0$. Assume that $B(\cdot)$ and $\alpha(\cdot)$ are independent. The Markov chain is observable and takes value in a set $\mathcal{M} = \{1, \ldots, M\}$. Denote $Q = (\lambda_{pq})_{p,q \in \mathcal{M}}$ the generator of $\alpha(\cdot)$ with $\lambda_{pq} \geq 0$ for $p \neq q$ and $\sum_{q \in \mathcal{M}} \lambda_{pq} = 0$ for each $p \in \mathcal{M}$. Let $\{\mathcal{F}_t\}_{t \geq 0}$ be the completed filtration generated by $B(\cdot)$ and $\alpha(\cdot)$ and augmented by the null sets.