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Conditional reachability of uncertain Max Plus Linear systems☆

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ABSTRACT

The reachability analysis problem of Max Plus Linear (MPL) systems has been properly solved using the Difference-Bound Matrices approach. In this work, the same approach is considered in order to solve the reachability analysis problem of MPL systems subjected to bounded noise, disturbances and/or modeling errors, called uncertain MPL (uMPL) systems. Moreover, using the results on the reachability analysis of uMPL systems, we solve the *conditional reachability problem*, herein defined as the support calculation of the probability density function involved in the stochastic filtering problem.

1. Introduction

Discrete event systems subject to only synchronization and time delay phenomena are a class of dynamic systems which can be described in a linear way in the max-plus algebra. The max-plus algebra is an idempotent semiring, an algebraic structure also called dioid (Baccelli, Cohen, Olsder, & Quadrat, 1992), in which the operations of sum ($\oplus$) and product ($\otimes$) are defined as the maximization and addition, respectively. Synchronization phenomena are modeled thanks to maximization: the start of a task waits for the completion of the preceding tasks, while the delay phenomena are depicted thanks to the classical sum: the completion time of a task is equal to the starting time plus the task duration. The Max Plus Linear (MPL) equations are used to model manufacturing systems, telecommunication networks, railway networks, and parallel computing (Baccelli et al., 1992; Brunsch, Raisch, & Hardouin, 2012). The linearity property has advantaged the emergence of a specific theory for the performance analysis (Heidergott, Olsder, & van der Woude, 2006) and the control of these systems, e.g., optimal open loop control (Cohen, Gaubert, & Quadrat, 1999; Lhommeau, Hardouin, Ferrier, & Ouerghi, 2005) and optimal state-feedback control. Among closed-loop strategies we can cite the model matching problem (Lhommeau, Hardouin, & Cottenceau, 2003) and the control strategies allowing the state to stay in a specific state subspace or semimodule (Amari, Demongodin, Loiseau, & Martinez, 2012; Gonçalves, Maia, & Hardouin, 2016; Katz, 2007; Maia, Hardouin, Santos-Mendes, & Loiseau, 2011; Necoara, De Schutter, van den Boom, & Hellendoorn, 2009).

The MPL systems may be subjected to noise and disturbances, which should be taken into account in order to avoid tracking error or closed loop instability (van den Boom & De Schutter, 2002). In general, these perturbations are max-plus-multiplicative and appear as uncertainties in the max plus model parameters. As a result the system matrices are uncertain. The Stochastic Max Plus Linear (SMPL) systems are defined as MPL systems where the matrices entries are characterized by random variables (DiLoreto, Gaubert, Katz, & Loiseau, 2010; Hardouin, Maia, Cottenceau, & Lhommeau, 2010; Heidergott, 2006; Olsder, Resing, Vries, Keane, & Hooghiemstra, 1990; van den Boom & De Schutter, 2002). In this work, although the probabilistic aspects of the uncertainties are not considered, we are interested in systems where the uncertain parameters can vary over a known interval. Formally, we define the uncertain Max-Plus Linear (uMPL) systems as *nondeterministic* MPL systems where, at each event step, the entries of the system

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matrices can, independently, take an arbitrary value within a real interval (Cechlrov, 2005; Cechlrov & Cuninghame-Green, 2002; Mykov, 2005).

To assess whether the system reaches a certain state from a set of initial conditions is of great interest in many applications and concerns the reachability analysis. In Gazarik, Michael, Kamen, and Edward (1999) residuation is used to determine if a state is reachable from a single initial condition. In Gaubert and Katz (2003), it is shown that if the initial set is a rational semimodule, the reachable set is also a rational semimodule. These authors mention that this set has a "simple shape" and suggest that an efficient numerical method remains to be designed. In Lu et al. (2012), reachability analysis of timed automata is tackled by considering max-plus polyhedra, a more general class of set than semimodules. For a more exhaustive presentation on max-plus polyhedra, see Allamigeon, Gaubert, and Goubault (2008). In Adzkiya, De Schutter, and Abate (2014b) the forward reachability problem for autonomous MPL systems is addressed by considering as initial set, the union of regions depicted as Difference Bound Matrices (DBM) (Dill, 1990). In Adzkiya, De Schutter, and Abate (2014a), backward reachability analysis has also been studied by considering the DBM approach. In Adzkiya, De Schutter, and Abate (2015), these results have been extended to nonautonomous MPL systems.

As shown in Adzkiya et al. (2014b), to describe an MPL system by means of DBM it is necessary to express it as a PieceWise Affine (PWA) system. This is always possible (Heemels, De Schutter, & Bemporad, 2001) and it is done by partitioning the state space into regions in which the system can be modeled by affine equations (in classical algebra). The PWA system is simply the union of these affine systems and the key point is that each affine system and its corresponding active state space region can be independently represented by one DBM. The main advantage of this representation is the existence of many efficient algorithms for DBM manipulation and its drawback is the upsizing of the representation of an MPL system from one compact state equation to multiple DBM. It should be remarked that, on one hand, Adzkiya , De Schutter et al. (2015) have proved that any region described as a max-plus polyhedron can also be described by a union of DBM. On the other hand, the complexity of the algorithms involving max-plus polyhedra are in general polynomial, while the complexity of the DBM approach critically depends on the number of PWA subsystems, which grows exponentially with the dimension of the system. Due to the exponential complexity, the DBM approach comfortably handles reachability computations for MPL models with *up to* twenty state variables, see Adzkiya , De Schutter et al. (2015, Sec. 5). Approaches based on max-plus polyhedra seems to be a promising way to reduce the complexity of reachability computations for MPL systems and, therefore, to extend the dimension of the addressable problem. However, to the best of the authors' knowledge, there are no approaches based on max-plus polyhedra for solving the forward and the backward reachability problem for general MPL systems and such methods remain to be designed.

In this work, we aim to extend the DBM approach in order to analyze uMPL systems. It is shown that uMPL systems can be partitioned into components that can be fully represented by DBM and that it is efficient for reachability analysis of uMPL systems. Then, for the forward reachability analysis, given a set of initial conditions represented by a union of finitely many DBM, the sets of states that *may be* reached at each event step are computed. Similarly, for the backward reachability analysis, given a set of final conditions represented by a union of finitely many DBM, the sets of all states that *may lead* to the set of final conditions in a fixed number of steps can be computed.

Bayesian methods provide a rigorous general framework for dynamic state estimation problems (Gordon, Salmond, & Smith, 1993). The objective of the Bayesian state estimation is to construct the posterior probability density function (PDF) of the states based on all information available. It should be noted that the computation of the states PDF is quite difficult. Although these problems are very closely related, this paper only concerns the reachability problem and therefore the purpose is limited to the calculation of the support of the prior and the posterior state estimation, which does not require the use of probability measures (Section 5). We define the *conditional reachability problem* as the support calculation of the posterior PDF of the uMPL system states. We assume that a sequence of measurements related to the state through an uMPL equation is given and then we show that this problem can be solved by using the results on reachability analysis of uMPL systems.

The paper is organized as follows: Section 2 recalls the MPL systems and their decompositions as PWA systems, as well as the DBM representation of PWA systems generated by MPL systems. Section 3 extends the DBM approach to uMPL systems. Section 4 presents reachability analysis for uMPL systems. Section 5 deals with the conditional reachability problem. Section 6 applies the results of the paper in order to solve the conditional reachability problem for a given uMPL system. Finally, Section 7 concludes the work.

2. Preliminaries

2.1. Max plus linear systems

A set S, endowed with two internal operations: *sum*($\oplus$) and *product*($\otimes$) is a **dioid** or **idempotent semiring** if the sum is associative, commutative and idempotent (i.e. $a \oplus a = a$) and the product is associative and left and right distributive with respect to the sum.[1] The null (or zero) element, denoted by ε, is such that $\forall a \in S, a \oplus \varepsilon = a$, and the identity element, denoted by e, is such that $\forall a \in S$, $a \otimes e = e \otimes a = a$. Besides, the zero element is absorbing for the $\otimes$ operation (i.e. $\forall a \in S$, $a \otimes \varepsilon = \varepsilon \otimes a = \varepsilon$) (Baccelli et al., 1992 Def. 4.1). In this algebraic structure, a partial order relation is defined by:

$$a \succeq b \Leftrightarrow a = a \oplus b. \tag{1}$$

Given these conditions, it appears that the set $\mathbb{R} \cup \{-\infty\} \cup \{\infty\}$ and the operations:$\alpha \oplus \beta \equiv \max\{\alpha, \beta\}$ and $\alpha \otimes \beta \equiv \alpha + \beta$, with $\varepsilon = -\infty$, $e = 0$, and with the convention that $\infty \otimes \varepsilon = \varepsilon$, is a dioid. Moreover, it can be stated that this is a **complete dioid** since it is closed for infinite sums and the left and right distributivity of the product extends to infinite sums[2] . This set is called **Max-Plus** semiring and noted by $\overline{\mathbb{R}}_{max}$. The $\oplus$ and $\otimes$ operations can be extended to matrices as follows. If $A, B \in \overline{\mathbb{R}}_{max}^{n\times p}$ and $C \in \overline{\mathbb{R}}_{max}^{p\times q}$, then: $[A \oplus B]_{ij} = a_{ij} \oplus b_{ij}$ and $[A \otimes C]_{ij} = \bigoplus_{k=1}^{p} a_{ik} \otimes c_{kj}$.

The autonomous model of an MPL system is given by:

$$\mathbf{x}(k) = A \otimes \mathbf{x}(k-1), \tag{2}$$

where the entries of matrix $A \in \overline{\mathbb{R}}_{max}^{n\times n}$ are the parameters of the model, a_{ij} represents the minimal delay between two events. The variable $k \in \mathbb{N}$ is an event-number and the state vector $\mathbf{x} \in \overline{\mathbb{R}}_{max}^{n}$ is a **dater**, i.e., $\mathbf{x}(k)$ contains the kth date of occurrence of each event of the system.

The nonautonomous model of an MPL system is defined as:

$$\mathbf{x}(k) = A \otimes \mathbf{x}(k-1) \oplus B \otimes \mathbf{u}(k), \tag{3}$$

where $\mathbf{u}$ is an external input and $B \in \overline{\mathbb{R}}_{max}^{n\times m}$.

Any nonautonomous MPL system can be transformed into an augmented autonomous MPL model by considering $F = (A\,B) \in \overline{\mathbb{R}}_{max}^{n\times(n+m)}$ and $\mathbf{y}(k-1) = \left(\mathbf{x}(k-1)^T\ \mathbf{u}(k)^T\right)^T$ (Baccelli et al., 1992 Sec. 2.5.4).

$$\mathbf{x}(k) = F \otimes \mathbf{y}(k-1). \tag{4}$$

[1] The product is not necessarily commutative.

[2] For complete dioids, the order relation (1) can be written as: $a \succeq b \Leftrightarrow a = a \oplus b \Leftrightarrow b = a \wedge b$, where $a \wedge b$ is the greatest lower bound of a and b.