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Brief paper

Stability analysis of semi-Markov switched stochastic systems ☆

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ABSTRACT

This paper investigates the asymptotic stability of semi-Markov switched stochastic systems. Based on the method of multiple Lyapunov functions and the structure of semi-Markov process, we provide sufficient conditions of stochastic asymptotic stability in the large for semi-Markov switched stochastic systems without the constraint of bounded transition rates. Particularly, our results generalize and improve some published results in the literature. An example and its simulation are given to illustrate the theoretical results.

1. Introduction

Markov switched systems can be used to model many practical systems subject to unpredictable changes, and the stability analysis of Markov switched systems has received considerable attentions, we refer the readers to Bolzern, Colaneri, and Nicolao (2006), Chatterjee and Liberzon (2006, 2008, 2012), Dang (2014), Deng, Luo, and Mao (2012), Faraji-Niri and Jahed-Motlagh (2016), Kao, Xie, Wang, and Karimi (2015), Kao, Zhu, and Qi (2015), Karimi (2011), Leth, Schioler, Gholami, and Cocquempot (2013), Mao and Yuan (2006), Shi and Li (2015), Tanelli, Picasso, Bolzern, and Colaneri (2010), Zhu (2014) and Zhu, Han, and Zhang (2012). In such systems, different subsystems are governed by a continuous time Markov process taking values in a finite state space, and therefore, the structure of Markov process plays an important role in the stability analysis of systems. The sojourn time at each state of Markov process follows exponential distribution, which leads to the transition rates between different states are constants. It is a key restriction because in many practical systems, such as fault tolerant control systems (see Johnson (1989)), the transition rates between different modes are time-varying, such a switching process cannot be modeled as a Markov process.

As a natural and important generalization of Markov process, the semi-Markov process (see Janssen and Manca (2006) and Kobayashi, Mark, and Turin (2012)) has time-varying transition rates. In recent years, some results on the stability analysis of the corresponding semi-Markov switched systems can be found in Hou, Luo, and Shi (2005), Hou, Luo, Shi, and Nguang (2006), Hou, Tong, and Zhang (2009), Huang and Shi (2011), Huang and Shi (2013), Li, Shi, and Wu (2017), Schioler, Simonsen, and Leth (2014) and Wei, Park, Karimi, Tian, and Jung (2017). For instance, by using the phase-type distribution to depict the sojourn time at each subsystem, Hou et al. (2005, 2006, 2009) studied the stochastic stability of semi-Markov jump systems. By using the Weibull distribution to depict the sojourn time at each subsystem, Huang and Shi (2011, 2013) studied the robust stochastic stability of semi-Markov switched systems with bounded transition rates. Schioler et al. (2014) studied the moment stability in the wide sense of semi-Markov switched deterministic systems without any reference to bounded transition rates. Schioler, Leth, Simonsen, and Khan (2015) considered the effect of random noise on the stochastic stability of semi-Markov switched systems and generalized the results of Schioler et al. (2014), but they assumed that the sojourn times of unstable subsystems should follow the exponential distributions (see Lemma 1 of Schioler et al. (2015)) and the sojourn times of all subsystems are upper bounded (see Lemma 4 of Schioler et al. (2015)). Wei et al. (2017) studied the stability and stabilization results for time-varying delay stochastic synchronization of semi-Markovian jump neutral networks without random noise disturbance. Li et al. (2017) presented the novel

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method and studied the stability analysis, robust controller design, robust filter design and fault detection for the semi-Markov jump systems in the deterministic case.

However, the above-mentioned works have imposed some very strict restrictions on the semi-Markov switched systems. For example, in Hou et al. (2005, 2006, 2009), Huang and Shi (2011, 2013) and Schioler et al. (2015), the sojourn time at each state of semi-Markov switching process was required to follow specific distribution; In Huang and Shi (2011, 2013), the generator matrix of semi-Markovian switching process was required to be bounded; In Li et al. (2017), Schioler et al. (2014) and Wei et al. (2017), all the subsystems were required to be deterministic, and so on.

Motivated by the foregoing discussion, in this paper we investigate the stochastic stability problem for the semi-Markov switched stochastic systems without the above mentioned restrictions. By utilizing the method of multiple Lyapunov functions and stochastic analysis theory, we obtain some sufficient conditions for stochastic asymptotic stability in the large of semi-Markov switched stochastic systems. Moreover, our results generalize and improve some published results in the literature, such as Theorem 3.2 in Zhu et al. (2012) and Theorem 5.37 in Mao and Yuan (2006).

The remainder of this paper is organized as follows. Section 2 describes some preliminaries. In Section 3, we obtain sufficient conditions of stochastic stability for semi-Markov switched stochastic systems via a set of inequalities based on the multiple Lyapunov functions and the structure of semi-Markov process. In Section 4, an example is given to illustrate the effectiveness of our results. Finally, the paper is concluded in Section 5.

Notation. Throughout this paper, $\mathbf{R}^n$ and $\mathbf{R}^{n\times m}$ denote, respectively, the n-dimensional Euclidean space and the set of $n \times m$ real matrices. $\mathbf{R}^+$ denotes the interval $[0, \infty)$, $|\cdot|$ denotes the absolute value in $\mathbf{R}$ and $\|\cdot\|$ denotes the Euclidean norm in $\mathbf{R}^n$. If A is a vector or matrix, its transpose is denoted by A^T. If A is a real symmetric matrix, $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ denote the smallest and largest eigenvalues, respectively. Let B_δ be the set of $\{x \in \mathbf{R}^n : \|x\| < \delta\}$. A function $\alpha(\cdot) : \mathbf{R}^+ \to \mathbf{R}^+$ is said to belong to class $\mathcal{K}$ if it is continuous, strictly increasing and $\alpha(0) = 0$, it is said to belong to class $\mathcal{K}_\infty$ if $\alpha(\cdot) \in \mathcal{K}$ and $\alpha(x) \to \infty$ as $x \to \infty$. If a, b are real numbers, then $a \vee b$ denotes the maximum of a and b, and $a \wedge b$ denotes the minimum of a and b. $I(\cdot)$ denotes the indicator function.

2. Preliminaries

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space. In this section, we first give the definition of semi-Markov process and some related notions.

Definition 1 (*See Hou et al. (2009)*)**.** Let $S = \{1, 2, \ldots, N\}$ be a finite state space. A stochastic process $\{r(t), t \geq 0\}$ is called a semi-Markov process on the probability space with finite state space S, if the following conditions hold.

1. The sample paths of $\{r(t), t \geq 0\}$ are right-continuous step functions and have left-handed limits with probability one.

2. Denote the kth jump point of the process $r(t)$ by T_k, $k = 0, 1, 2, \ldots$, where $t_0 = T_0 < T_1 < T_2 < \cdots < T_k < \cdots$, $T_k \uparrow +\infty$, and the process $r(t)$ possesses Markov property at each T_k, $k = 0, 1, 2, \ldots$.

3. $F_{ij}(t) := P(T_{k+1} - T_k \leq t | r(T_k) = i, r(T_{k+1}) = j) = F_i(t)(i, j \in S, t \geq 0)$ does not depend on j and k.

Let $\{N_r(t), t \geq 0\}$ be the number of switches of $r(t)$ on the interval $(t_0, t]$. Obviously, for any $t \geq t_0$, $k \geq 0$, $N_r(t) = k$ is equivalent to $t \in [T_k, T_{k+1})$, $T_{k+1} - T_k$ is the kth sojourn time. Let τ_i be the sojourn time in state $i \in S$. By Definition 1, the structure of semi-Markov process $\{r(t), t \geq 0\}$ can be characterized by the following two notions (see p. 455 of Kobayashi et al. (2012)):

1. The transition probability matrix

$$P_{N\times N} = (p_{ij})_{N\times N}, \ \forall i, j \in S, \tag{1}$$

where $p_{ij} = P(r(T_{k+1}) = j | r(T_k) = i)$ is the probability with which the process makes a transition from state i to state j at time T_{k+1}, $k \geq 0$.

2. The set of distribution functions of sojourn times τ_i, $i \in S$,

$$\begin{aligned} F_i(t) &:= P(\tau_i \leq t) \\ &= P(T_{k+1} - T_k \leq t | r(T_k) = i), \ \forall k \geq 0, \end{aligned} \tag{2}$$

where $F_i(t)$ has continuous differentiable density $f_i(t)$.

Throughout the paper, we assume that the switching process $r(t)$ satisfies the following conditions.

Assumption 1 (*See Definition 4.2 of Chatterjee and Liberzon (2008)*)**.** 1. The sequence $\{T_{k+1} - T_k, k \geq 0\}$ is a collection of independent random variables with $E(T_{k+1} - T_k) < \infty$.

2. The sequence $\{r(T_k), k \geq 0\}$ is a discrete-time Markov chain with transition probability matrix $P = (p_{ij})_{N\times N}$.

3. The sequence $\{T_{k+1} - T_k, k \geq 0\}$ is independent of $\{r(T_k), k \geq 0\}$.

Remark 1. Definition 4.2 of Chatterjee and Liberzon (2008) gave the definition of class GH, which requires the switching process $r(t)$ to satisfy not only the items 1, 2, 3 of Assumption 1, but also the condition: τ_i, $i \in S$ are identical distributed. Although it is not the most general case, the class GH is a standard hypothesis for semi-Markov processes. Obviously, Assumption 1 generalizes the class GH to a certain extent.

Next, we give the notion of generator matrix of semi-Markov process $\{r(t), t \geq 0\}$. For arbitrary $t \geq 0$, let $h(t) := t - \sup\{T_k : T_k \leq t, k \geq 0\}$. A simple calculation shows that the transition rates

$$\begin{aligned} q_{ij}(h) &:= \lim_{\Delta t\to 0} \frac{P(r(t+\Delta t) = j | r(t) = i)}{\Delta t} \\ &= p_{ij}\frac{f_i(h)}{1 - F_i(h)}, \quad \forall j \neq i \in S, \end{aligned} \tag{3}$$

from state i to another state $j(\neq i)$, and

$$q_{ii}(h) := -\sum_{j\in S, j\neq i} q_{ij}(h), \ \forall i \in S. \tag{4}$$

Thus, we get the generator matrix

$$\Lambda(h) := (q_{ij}(h))_{N\times N}, \quad h \geq 0, \tag{5}$$

which governs the evolution of semi-Markov process $\{r(t), t \geq 0\}$.

In this paper, we consider the following semi-Markov switched stochastic system of the form

$$\begin{aligned} &dx(t) = f(x(t), r(t))dt + g(x(t), r(t))dB(t), \\ &x(t_0) = x_0 \in \mathbf{R}^n, r(t_0) = r_0 \in S, \end{aligned} \tag{6}$$

where $\{r(t), t \geq 0\}$ is a semi-Markov process, $\{B(t), t \geq 0\}$ is a d-dimensional Brownian motion. For convenience, we let the initial values x_0 and r_0 be non-random, namely $x_0 \in \mathbf{R}^n$ and $r_0 \in S$. As a usual, we assume that $B(t)$ and $r(t)$ are independent. $f(\cdot,\cdot) : \mathbf{R}^n \times S \mapsto \mathbf{R}^n$ and $g(\cdot,\cdot) : \mathbf{R}^n \times S \mapsto \mathbf{R}^{n\times d}$. Both f and g satisfy the local Lipschitz condition and the linear growth condition. Obviously, these conditions can ensure that system (6) has a unique solution, and we denote it by $x(t; t_0, x_0, r_0)$ with initial condition $x(t_0) = x_0$ and $r(t_0) = r_0$, or $x(t)$ for simplicity. We also assume that $f(0, i) = 0$, $g(0, i) = 0$ for each $i \in S$. This implies that system (6) admits a trivial solution $x(t) \equiv 0$.