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Brief paper

On a class of generating vector fields for the extremum seeking problem: Lie bracket approximation and stability properties[☆]

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ABSTRACT

In this paper, we describe a broad class of control functions for extremum seeking problems. We show that it unifies and generalizes existing extremum seeking strategies which are based on Lie bracket approximations, and allows to design new controls with favorable properties in extremum seeking and vibrational stabilization tasks. The second result of this paper is a novel approach for studying the asymptotic behavior of extremum seeking systems. It provides a constructive procedure for defining frequencies of control functions to ensure the practical asymptotic and exponential stability. In contrast to many known results, we also prove asymptotic and exponential stability in the sense of Lyapunov for the proposed class of extremum seeking systems under appropriate assumptions on the vector fields.

1. Introduction

In many control applications, the goal is to operate a system in some optimal fashion. Often, however, the optimal operating point is unknown or may even change over time so that it cannot be determined a priori. Extremum seeking control is a control methodology to solve such problems of stabilizing and tracking an a priori unknown optimal operating point. Typically, it is model-free and minimizes or maximizes the steady-state map of a system. The steady-state map maps constant control input values to the steady-state output values. It is a well-defined map under appropriate assumptions on the system. There exist many ways to design the extremum seeking strategies. A classical perturbation-based approach is to use the controls consisting of time-periodic oscillating inputs (often called dither, excitation, perturbation or learning signal) and state-dependent vector fields in order to gather information about the unknown steady-state map. Based on the perturbed input and the perturbed output response, typically the gradient or other descent directions of the steady-state map are approximated or estimated by appropriate signal processing or filtering methods, see, e.g. Dürr, Krstić, Scheinker, and Ebenbauer (2017), Dürr, Stanković, Ebenbauer, and Johansson (2013), Gelbert, Moeck, Paschereit, and King (2012), Guay and Dochain (2015), Guay and Zhang (2003), Haring and Johansen (2017), Krstić and Ariyur (2003), Krstić and Wang (2000), Nešić, Tan, Moase, and Manzie (2010) and Tan, Moase, Manzie, Nešić, and Mareels (2010). Hereby, the shape of control functions plays an important role since it influences the speed of convergence and may be subject to input constraints. In the literature, different types of excitation signals have been analyzed, see, e.g. Chioua, Srinivasan, Perrier, and Guay (2010), Nešić (2009), Scheinker and Krstić (2014a), Scheinker and Scheinker (2016) and Tan, Nešić, and Mareels (2008).

In this paper, we propose a novel class of vector fields for extremum seeking controls based on Lie bracket approximation techniques (Dürr et al., 2017, 2013). The first contribution of this paper is a formula describing a whole class of vector fields for an extremum seeking system which allows to approximate a gradient flow in various ways. The formula unifies and generalizes previously known controls presented in Dürr et al. (2013), Scheinker and Krstić (2014a), Scheinker and Scheinker (2016) and Suttner and Dashkovskiy (2017) and allows to generate new extremum seeking strategies with desirable properties. In particular, we demonstrate benefits of this formula by designing a control which has bounded update rates *and* vanishing amplitudes at the same time.

Moreover, the second contribution is a rigorous proof of the asymptotic and exponential stability *in the sense of Lyapunov*, under appropriate assumptions on the considered class of generating

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vector fields. This is in contrast to many results in the literature, where typically *practical* stability results are established. The proof also extends the techniques developed in Grushkovskaya and Zuyev (2013, 2014), Zuyev (2016) and Zuyev, Grushkovskaya, and Benner (2016), to a wide class of cost functions and to systems whose vector fields are non-differentiable at the origin. An advantage of these techniques is the possibility to estimate the decay rate of solutions of the extremum seeking systems.

Finally, we demonstrate that the proposed formula is not only of use in extremum seeking but also in vibrational stabilization problems (Michalowsky & Ebenbauer, 2013; Scheinker & Krstić, 2013).

The paper is organized as follows. Section 2 contains some preliminary results on extremum seeking based on Lie bracket approximations. In Section 3, we present a novel formula to approximate the gradient flows and establish various asymptotic stability conditions. In Section 4, we illustrate several extremum seeking strategies by using numerical simulations, and discuss the application of the obtained results to the vibration stabilization problem. The Appendix contains auxiliary lemmas and proofs.

2. Preliminaries

2.1. Notations

Throughout the text, $\mathbb{R}^+$ denotes the set of all non-negative real numbers, $B_\delta(x^*)$ is the δ-neighborhood of $x^* \in \mathbb{R}^n$, $\overline{B_\delta(x^*)}$ is its closure. For $h \in C^1(\mathbb{R}^n;\mathbb{R})$, $\xi \in \mathbb{R}^n$, we define the column $\nabla h(\xi) := \frac{\partial h(x)}{\partial x}^T\Big|_{x=\xi}$. For a function $f : \mathbb{R} \to \mathbb{R}$, $f(z) = O(z)$ as $z \to 0$ means that there is a $c > 0$ such that $|f(z)| \le c|z|$ in some neighborhood of 0. For $f, g : \mathbb{R}^n \to \mathbb{R}^n$, $x \in \mathbb{R}^n$, we denote the Lie derivative as $L_g f(x) = \lim_{s\to 0} \frac{f(x+sg(x))-f(x)}{s}$, and $[f, g](x) = L_f g(x) - L_g f(x)$ is the Lie bracket. For $m, n \in \mathbb{Z}$, the notation $i = \overline{m, n}$ means that $i = m, m + 1, \dots, n$. For $a, b \in \mathbb{R}^n$, we denote their open convex hull as $\mathrm{co}\{a, b\} = \{\lambda a + (1 - \lambda)b \mid \lambda \in (0, 1)\}$.

2.2. Lie bracket approximations

Consider a control-affine system

$$\dot{x} = f_0(x) + \sum_{j=1}^{\ell} f_j(x)\sqrt{\omega}u_j(\omega t), \tag{1}$$

where $x = (x_1, \dots, x_n)^T \in \mathbb{R}^n$, $x(t_0) = x^0 \in \mathbb{R}^n$ (without loss of generality, we assume $t_0 = 0$), $\omega > 0$, $f_j : \mathbb{R}^n \to \mathbb{R}^n$, $j = \overline{1, \ell}$. Assume that:

A0 $u_j(t)$ are continuous T-periodic functions, $\int_0^T u_j(\tau)d\tau = 0$, $\int_0^T \int_0^\theta u_i(\theta)u_j(\tau)d\tau ds = \beta_{i,j}T$, $T > 0$, $\beta_{i,j} \in \mathbb{R}$, $i, j = \overline{1, \ell}$.

It can be shown that the trajectories of (1) approximate trajectories of the following *Lie bracket system*:

$$\dot{\bar{x}} = f_0(\bar{x}) + \sum_{i<j} \beta_{j,i}[f_i, f_j](\bar{x}), \quad \bar{x}(0) = x^0. \tag{2}$$

The stability properties of systems (1) and (2) are related as follows.

Lemma 1 (Dürr et al., 2013). *Let $f_0, f_i \in C^2(\mathbb{R}^n;\mathbb{R}^n)$, and u_i satisfy A0, $i = \overline{1, \ell}$. If a compact set $S \subset \mathbb{R}^n$ is locally (globally) uniformly asymptotically stable for (2) then it is locally (semi-globally) practically uniformly asymptotically stable for (1).*

Below we recall the notion of practical stability.

Definition 1. A compact set $S \subset \mathbb{R}^n$ is said to be locally practically uniformly asymptotically stable for (1) if:

– it is practically uniformly stable, i.e. for every $\varepsilon > 0$ there exist $\delta > 0$ and $\omega_0 > 0$ such that, for all $t_0 \ge 0$ and $\omega > \omega_0$, if $x^0 \in B_\delta(S)$ then the corresponding solution of (1) satisfies $x(t) \in B_\varepsilon(S)$ for all $t \ge t_0$;

– $\hat{\delta}$-practically uniformly attractive with some $\hat{\delta} > 0$, i.e. for every $\varepsilon > 0$ there exist $t_1 \ge 0$ and $\omega_0 > 0$ such that, for all $t_0 \ge 0$ and $\omega > \omega_0$, if $x^0 \in B_{\hat{\delta}}(S)$ then the corresponding solution of (1) satisfies $x(t) \in B_\varepsilon(S)$ for all $t \ge t_0 + t_1$.

If the attractivity property holds for every $\hat{\delta} > 0$, then the set S is called semi-globally practically uniformly asymptotically stable for (1).

2.3. Extremum seeking problem

In this paper, we address a class of extremum seeking problems related to the unconstrained minimization of a cost function J. We assume that $J \in C^2(\mathbb{R}^n;\mathbb{R})$ is unknown (as an analytic expression) but can be evaluated (measured) at each $x \in \mathbb{R}^n$. The goal is to construct a control system of the form $\dot{x} = u(t, J(x))$ such that the (local) minima of J have some desired stability properties for this system. In this setup, a static map J corresponds to the steady-state map of a system. However, the extremum seeking based on Lie bracket approximations can be applied to much more general scenarios, including dynamic maps (dynamical systems), constrained optimization problems, distributed and multi-agent extremum seeking, stabilization, synchronization and consensus problems as well as problems on manifolds, etc. The results obtained in this paper can be applied to such more general problems but are not discussed here for the sake of simplicity.

The underlying idea of the extremum seeking based on the Lie bracket approximations is as follows. Suppose that $n = 1$, i.e. $x \in \mathbb{R}$, and consider the system

$$\dot{x} = 2J(x)\sqrt{\omega}\cos(\omega t) + \sqrt{\omega}\sin(\omega t). \tag{3}$$

It can be seen that the Lie bracket system for (3) represents the gradient flow of J:

$$\dot{\bar{x}} = \tfrac{1}{2}[2J(\bar{x}), 1] = -\nabla J(\bar{x}). \tag{4}$$

Thus, the trajectories of system (3) approximate trajectories of the gradient flow of J and they converge, for example, if J is convex and has minima, into an arbitrary small neighborhood of the set of minima of J, for sufficiently large ω. For $n > 1$, the gradient flow can be approximated in a similar way, see Dürr et al. (2013) for details.

3. Main results

3.1. Vector fields for approximating gradient flows

Observe that there are many ways to define the vector fields of system (1) such that the corresponding Lie bracket system has the form (4). For example, consider the system

$$\dot{x} = e^{J(x)}\sqrt{\omega}\cos(\omega t) + e^{-J(x)}\sqrt{\omega}\sin(\omega t). \tag{5}$$

Computing $\left[e^{J(x)}, e^{-J(x)}\right]$ yields $-2\nabla J(x)$ and, hence, the associated Lie bracket system is again of the form (4).

The main idea and the first main result of this paper is the description of a class of vector fields for system (1) such that the corresponding Lie bracket system (2) represents a gradient-like flow of J. Consider first the system

$$\dot{x} = F_1(J(x))\sqrt{\omega}u_1(\omega t) + F_2(J(x))\sqrt{\omega}u_2(\omega t). \tag{6}$$

We begin with the one-dimensional case $x \in \mathbb{R}$ to simplify the presentation, and the multi-dimensional case will be considered later as an extension.