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Brief paper

Output feedback control of general linear heterodirectional hyperbolic ODE–PDE–ODE systems[☆]

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ABSTRACT

This paper considers the backstepping design of observer-based compensators for general linear heterodirectional hyperbolic ODE–PDE–ODE systems, where the ODEs are coupled to the PDEs at both boundaries and the input appears in an ODE. A state feedback controller is designed by mapping the closed-loop system into a stable ODE–PDE–ODE cascade. This is achieved by representing the ODE at the actuated boundary in Byrnes–Isidori normal form. The resulting state feedback is implemented by an observer for a collocated measurement of the PDE state, for which a systematic backstepping approach is presented. The exponential stability of the closed-loop system is verified in the ∞-norm. It is shown that all design equations can be traced back to kernel equations known from the literature, to simple Volterra integral equations of the second kind and to explicitly solvable boundary value problems. This leads to a systematic approach for the boundary stabilization of the considered class of ODE–PDE–ODE systems by output feedback control. The results of the paper are illustrated by a numerical example.

1. Introduction

In recent years, the *backstepping approach* (see, e.g., Krstic & Smyshlyaev, 2008) was utilized to provide systematic solutions for the boundary stabilization of PDE–ODE systems. At first, *PDE–ODE cascades* were considered, where an ODE is coupled to a PDE or vice versa (see, e.g., Krstic, 2009). A more challenging problem is the stabilization of *coupled PDE–ODE systems*, which arise directly in the modelling if dynamic boundary conditions (BCs) have to be taken into account (see, e.g., Sagert, Di Meglio, Krstic, & Rouchon, 2013, Tang & Xie, 2011). Recently, the extension of the previous backstepping results to coupled PDE–ODE systems, where the PDEs describe a general heterodirectional hyperbolic system, attracted the attention of many researchers. The interest in this problem stems from applications including coupled string networks (see, e.g., Ch. 6 Luo, Guo, & Morgul, 1999), networks of open channels and transmission lines (see, e.g., Bastin & Coron, 2016). By making use of the results in Hu, Meglio, Vazquez, and Krstic (2016), a backstepping approach for this system class with constant coefficients was presented in Di Meglio, Argomedo, Hu, and Krstic (2018). Subsequently, the work (Deutscher, Gehring, & Kern, 2018) considered the case of spatially-varying coefficients by making use of the results in Hu, Vazquez, Meglio, and Krstic (2015). The approach in Di Meglio et al. (2018) only treats the state feedback controller design, while in Deutscher et al. (2018) also an observer for an anticollocated measurement was presented in order to design an observer-based compensator. For enlarging the applicability of the backstepping method, it is reasonable to consider *ODE–PDE–ODE systems*, where the actuated boundary is described by an ODE with an input. The latter describes the dynamics of the actuator and thus leads to a much more involved stabilization problem for the underlying distributed-parameter system (DPS). A first solution of this problem for a 2×2 hyperbolic system with a fully actuated ODE at $z = 1$ is presented in Bou Saba, Bribiesca-Argomedo, Di Loreto, and Eberard (2017). Therein, a single transformation is proposed to map the system into a cascade of an ODE and a coupled PDE–ODE system. The backstepping transformation follows from a new type of kernel equations, that consists of PDEs coupled with ODEs. Furthermore, exponential stability of the coupled target system is shown in the L_2 sense.

In this paper, the design of observer-based compensators for general linear heterodirectional ODE–PDE–ODE systems with spatially-varying coefficients is considered. The input of the system acts on the ODE appearing at $z = 1$. Furthermore, a collocated boundary measurement of the distributed state is assumed for

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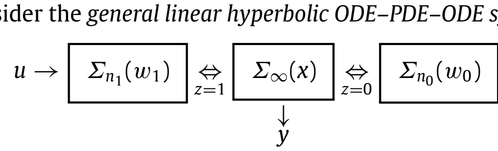
the observer design. In order to solve the corresponding output feedback stabilization problem, the results in Deutscher et al. (2018) are generalized. Therein, a *two-step approach* is utilized to determine the controller for the coupled PDE–ODE system. In the first step, the DPS is mapped into backstepping coordinates. As the related target system is of much simpler structure, this significantly facilitates the decoupling into a stable PDE–ODE cascade in the second step. In the paper this method is applied to the design of the state feedback controller in order to map the closed-loop system into an ODE–PDE–ODE system, where the PDE subsystem is decoupled from the ODE at the unactuated boundary. The corresponding transformations can directly be obtained by solving the kernel equations in Hu et al. (2015) and simple Volterra integral equations of the second kind. As the input acts on the ODE, the last step for obtaining a stable *ODE–PDE–ODE cascade* requires the introduction of new coordinates to represent the ODE at $z = 1$ in its multivariable *Byrnes–Isidori normal form* (see, e.g., Isidori, 1995, Ch. 5.1). For this, a *vector relative degree* of one is assumed, which is a requirement often met in applications. Typical examples are hyperbolic flexible structures, where a rigid body is attached at the actuated boundary. This assumption also includes the full actuation considered in Bou Saba et al. (2017) as a special case. On the basis of the resulting Byrnes–Isidori normal form, the state feedback controller can easily be determined. For its implementation, a collocated observer is designed. Compared to an anticollocated observer, this is a much more challenging problem as the ODE at the unactuated boundary of the observer is, in this case, subject to a coupling with the PDEs from both boundaries. By making use of the two-step approach, it is shown that only the usual observer kernel equations for the PDE subsystem and simple Volterra integral equations of the second kind have to be solved for the observer design. The solutions of all other design equations are obtained explicitly. The separation principle is verified for the corresponding closed-loop system. This is possible by utilizing the simple structure of the target systems in order to calculate the closed-loop solution pointwise in space. On the basis of this result, the exponential decay of the distributed closed-loop states w.r.t. the ∞-norm, i.e., pointwise in space, is shown. This leads to the systematic design of observer-based compensators for a large class of coupled hyperbolic PDEs with dynamic BCs at both boundaries.

The next section presents the formulation of the considered output feedback stabilization problem. In Section 3, the state feedback is designed. In order to implement this controller, Section 4 considers the observer design for a collocated measurement. Section 5 is devoted to the stability analysis of the closed-loop system with the observer-based compensator. The results of the paper are illustrated by means of a numerical example.

2. Problem formulation

Consider the *general linear hyperbolic ODE–PDE–ODE system*



described by

$$\partial_t x(z,t) = \Lambda(z)\partial_z x(z,t) + A(z)x(z,t) \tag{1a}$$
$$x_2(0,t) = Q_0 x_1(0,t) + C_0 w_0(t), \quad t > 0 \tag{1b}$$
$$x_1(1,t) = Q_1 x_2(1,t) + C_1 w_1(t), \quad t > 0 \tag{1c}$$
$$\dot{w}_0(t) = F_0 w_0(t) + B_0 x_1(0,t), \quad t > 0 \tag{1d}$$
$$\dot{w}_1(t) = F_1 w_1(t) + B_1 x_2(1,t) + Bu(t), \quad t > 0 \tag{1e}$$
$$y(t) = x_2(1,t), \quad t \geq 0 \tag{1f}$$

that consists of n coupled *transport PDEs* (1a) with the distributed state $x(z,t) = [x^1(z,t) \quad \dots \quad x^n(z,t)]^\top \in \mathbb{R}^n$ and the ODEs (1d) and (1e) with the lumped states $w_0(t) \in \mathbb{R}^{n_0}$ and $w_1(t) \in \mathbb{R}^{n_1}$. The input is $u(t) \in \mathbb{R}^p$ and the *collocated measurement* is $y(t) \in \mathbb{R}^m$ with $p + m = n$ and $p, m \geq 1$. Furthermore, $\Lambda(z)$ in (1a) is given by

$$\Lambda(z) = \operatorname{diag}(\lambda_1(z), \dots, \lambda_n(z)), \tag{2}$$

where $\lambda_i \in C^1[0,1]$, $i = 1, 2, \dots, n$, and $\lambda_1(z) > \dots > \lambda_p(z) > 0 > \lambda_{p+1}(z) > \dots > \lambda_n(z)$, $z \in [0,1]$. Moreover, the matrix $A(z) = [A_{ij}(z)]$ in (1a) satisfies $A_{ij} \in C^1[0,1]$, $i, j = 1, 2, \dots, n$ and $A_{ii}(z) = 0$, $z \in [0,1]$, $i = 1, 2, \dots, n$. Note, that the latter condition means no loss of generality (see, e.g., Hu et al., 2016). The *initial conditions (ICs)* of (1) are $x(z,0) = x_0(z) \in \mathbb{R}^n$, $z \in [0,1]$, $w_0(0) = w_{0,0} \in \mathbb{R}^{n_0}$ and $w_1(0) = w_{1,0} \in \mathbb{R}^{n_1}$.

With the matrices

$$E_1 = \begin{bmatrix} I_p \\ 0 \end{bmatrix} \in \mathbb{R}^{n \times p} \quad \text{and} \quad E_2 = \begin{bmatrix} 0 \\ I_m \end{bmatrix} \in \mathbb{R}^{n \times m}, \tag{3}$$

the transport in the negative direction of the spatial coordinate z is described by the states $x_1(z,t) = E_1^\top x(z,t) \in \mathbb{R}^p$ while the remaining states $x_2(z,t) = E_2^\top x(z,t) \in \mathbb{R}^m$ take the transport in the opposite direction into account. This gives rise to the state partitioning $x(z,t) = \operatorname{col}(x_1(z,t), x_2(z,t))$ for the PDE subsystem (1a)–(1c). The PDEs for the states x_1 are defined on $(z,t) \in [0,1) \times \mathbb{R}^+$, while the PDEs for the states x_2 evolve on $(z,t) \in (0,1] \times \mathbb{R}^+$. Hence, the distributed-parameter subsystem (1a)–(1c) is a *heterodirectional system* (see Hu et al., 2016). The following assumptions are imposed:

(A1) (F_0, B_0) is controllable,
(A2) (C_0, F_0) is observable,
(A3) $\operatorname{rank} C_1 = \operatorname{rank} B = p$, $n_1 \geq p$ and
(A4) $\det(C_1 B) \neq 0$.

The Assumptions (A1) and (A2) are needed for the stabilization of the ODE subsystems appearing in the state feedback and observer design. Moreover, Assumptions (A3) and (A4) are introduced so that a Byrnes–Isidori normal form exists for (C_1, F_1, B_1) (see Section 3.2) and $n_1 \geq p$ is assumed in order to avoid overactuation, i.e., more inputs than states. Finally, (A4) means that (C_1, F_1, B) has a *vector relative degree* equal to one.

This paper is concerned with the *backstepping design* of an *observer-based compensator*, that stabilizes the system (1).

3. State feedback design

In what follows the *state feedback controller*

$$u(t) = \mathcal{K}[w_0(t), w_1(t), x(t)] \tag{4}$$

with the formal *feedback operator* $\mathcal{K}$ is determined by mapping (1) into a stable *ODE–PDE–ODE cascade*.

3.1. Decoupling of the PDE subsystem

It is shown in Deutscher et al. (2018) that the backstepping transformation and the transformation to decouple the PDE subsystem from the w_0-system at $z = 0$

$$\tilde{x}(z,t) = x(z,t) - \int_0^z K(z,\zeta)x(\zeta,t)\mathrm{d}\zeta = \mathcal{T}_1[x(t)](z) \tag{5a}$$

$$\tilde{x}(z,t) = \mathcal{T}_2^{-1}[\vartheta(t)](z) + N_I(z)w_0(t) \tag{5b}$$

with

$$\mathcal{T}_2^{-1}[\vartheta(t)](z) = \vartheta(z,t) + \int_0^z P_I(z,\zeta)\vartheta(\zeta,t)\mathrm{d}\zeta \tag{6}$$