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Brief paper

Generalized reciprocally convex combination lemmas and its application to time-delay systems☆

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ABSTRACT

Various efficient matrix inequalities have recently been proposed to deal with the stability analysis of linear systems with time-varying delays. This paper provides more insights on the relationship between some of them. We present an equivalent formulation of Moon et al.'s inequality, allowing us to discover strong links not only with the most recent and efficient matrix inequalities such as the reciprocally convex combination lemma and also its relaxed version but also with some previous inequalities such as the approximation inequality introduced in Shao (2009) or free-matrix-based inequality. More especially, it is proved that these existing inequalities can be captured as particular cases of Moon et al.'s inequality. Examples show the best tradeoff between the reduction of conservatism and the numerical complexity.

1. Introduction

Providing less conservative and computationally efficient stability conditions for linear systems subject to time-varying delays has attracted considerable attention over the past decades. To deal with integral quadratic terms that arise from the derivative of Lyapunov–Krasovskii functional, two main technical steps, namely the derivation of efficient integral and matrix inequalities are usually adopted. Several attempts have been done concerning integral inequalities such as Jensen (Fridman, 2014), Wirtinger-based (Seuret & Gouaisbaut, 2013), auxiliary-based (Hien & Trinh, 2015; Park, Lee, & Lee, 2015), Bessel inequalities (Seuret & Gouaisbaut, 2015) or polynomials-based inequality (Lee, Lee, & Park, 2017). Although these inequalities have shown a great interest for constant delay systems, their application to time/fast-varying delays reveals additional difficulties related to the non-convexity of the resulting terms. Therefore, some matrix inequalities are employed to address this problem and to derive convex conditions. A huge number of papers have studied the ways to combine efficient integral and matrix inequalities. The reader may look for instance to Liu and Seuret (2017) and Zhang, He, Jiang, and Wu (2017). Hence, a first method corresponds to the employment of Young's or Moon et al.'s inequalities (Moon, Park, Kwon, & Lee, 2001), after the application of an integral inequality. It is also noted that the recent free-matrix-based inequality (Zeng, He, Wu, & She, 2015) can be interpreted as the merge of the Wirtinger-based inequality and Moon et al.'s inequality. Furthermore, the reciprocally convex lemma was proposed in Park, Ko, and Jeong (2011). The novelty of this method consists in merging the non-convex terms into a single expression to derive an accurate convex inequality. It was notably shown in Liu and Seuret (2017) that the reciprocally convex combination lemma (Park et al., 2011) leads to the same conservatism as the Moon et al.'s inequality when considering Jensen-based stability criteria, but with a lower computational burden. More recently, a relaxed reciprocally convex combination lemma was developed in Zhang, He, Jiang, Wu, and Zeng (2016) without requiring any extra decision variable. This inequality was recently extended by the same authors in Zhang, He, Jiang, Wu, and Wang (2017).

The present paper aims at providing more insights on the relationship between some of these bounding methods. We present an equivalent formulation of Moon et al.'s inequality, allowing us to discover strong links not only with the most recent and efficient matrix inequalities such as the reciprocally convex combination lemma (Park et al., 2011) or its relaxed version (Zhang et al., 2016) but also with some previous inequalities such as Shao (2009) and Zeng et al. (2015). More especially, we prove that these existing inequalities can be captured as particular cases of Moon et al.'s

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inequality. Examples show the best tradeoff between the reduction of conservatism and the numerical complexity.

Notations: Throughout the paper, $\mathbb{R}^n$ denotes the n-dimensional Euclidean space. The notations $\mathbb{R}^{n\times m}$ and $\mathbb{S}^n$ are the set of $n\times m$ real matrices and of $n\times n$ real symmetric matrices, respectively. The notation $P\in\mathbb{S}^n_+$, means that $P\in\mathbb{S}^n$ and $P\succ 0$, which means that P is symmetric positive definite. The symmetric elements of a symmetric matrix will be denoted by $*$. For any matrices A, B of appropriate dimension, the matrix $diag(A,B)$ stands for $\left[\begin{smallmatrix}A&0\\0&B\end{smallmatrix}\right]$. The matrices I_n and $0_{n,m}$ represent the identity and null matrices of appropriate dimension and, when no confusion is possible, the subscript will be omitted. Moreover, for any square matrix A, we define $\mathrm{He}(A)=A+A^\top$. For any $h>0$ and any function $x:[-h,+\infty)\to\mathbb{R}^n$, the notation $x_t(\theta)$ stands for $x(t+\theta)$, for all $t\geq 0$ and all $\theta\in[-h,0]$.

2. Matrix inequalities for systems with time-varying delays

When considering stability of systems with time-varying delays, the problem often relies on finding a lower bound of a reciprocally convex quadratic term Θ given by

$$\Theta(\alpha)=\Gamma^\top\begin{bmatrix}\frac{1}{\alpha}R & 0\\ 0 & \frac{1}{1-\alpha}R\end{bmatrix}\Gamma,\quad \forall\alpha\in(0,1), \tag{1}$$

where, for given integers n and m such that $2n\leq m$, R is in $\mathbb{S}^n_+$, Γ in $\mathbb{R}^{2n\times m}$, such that $rank(\Gamma)=2n$. There are two main methods to find lower bounds. The first one is based on the Moon et al.'s inequality (see, e.g., the survey paper (Xu & Lam, 2008)). The second method is the so-called reciprocally convex combination lemma developed in Park et al. (2011). The conservatism induced by these two inequalities are independent. While, in some cases, such as stability conditions resulting from the application of the Jensen inequality, these two methods lead to equivalent results on examples (Liu & Seuret, 2017). In general, reciprocally convex combination lemma is more conservative than Moon et al.'s inequality (see e.g., Liu & Seuret, 2017; Zeng et al., 2015).

The objective of this paper is to provide more insights on the relationship between these two classes of bounding methods. In particular, we show that the Moon et al.'s inequality encompasses the reciprocally convex lemmas as particular cases. Moreover, following this idea, we propose a generalization of reciprocally convex lemmas, which again represents a particular case of the Moon et al.'s inequality. This generalization allows providing less restrictive results than the recent extension of the reciprocally convex lemmas (Zhang, He, Jiang, Wu et al., 2017, 2016; Zhang, Han, Seuret, & Gouaisbaut, 2017).

3. Modified Moon et al. 's inequality

3.1. Main result

The main result of the paper is stated below. It corresponds to a method to obtain a lower bound of the matrix Θ defined above, based on Moon et al.'s inequality, which is recalled below.

Lemma 1 (*Xu & Lam, 2008*). *For any $x,y\in\mathbb{R}^n$, any scalar $\epsilon>0$, any matrix R in $\mathbb{S}^n_+$, the following inequality holds*

$$2x^Ty\leq\epsilon^{-1}x^TRx+\epsilon y^TR^{-1}y.$$

The relationship between Moon et al.'s inequality and the reciprocally convex combination lemma has already studied in Liu and Seuret (2017). The next lemma will extend this work and formulate a generalization of the reciprocally convex combination lemma. The main result of this paper is stated as follows:

Lemma 2. *Consider a parameter dependent matrix $\Phi(\alpha)$ in $\mathbb{S}^m$, such that the convex inequality*

$$\Phi(\alpha)\preceq(1-\alpha)\Phi(0)+\alpha\Phi(1) \tag{2}$$

holds for all α in $[0,1]$. If there exist a matrix R in $\mathbb{S}^n_+$ and two matrices N_1, N_2 in $\mathbb{R}^{m\times n}$ such that the inequality

$$\Psi(\alpha)=\begin{bmatrix}\Phi(\alpha)-\Gamma^\top\mathcal{R}^0(\alpha)\Gamma-\mathrm{He}\left(\Gamma^\top\begin{bmatrix}(1-\alpha)N_1^\top\\ \alpha N_2^\top\end{bmatrix}\right) & *\\ \alpha N_1^\top+(1-\alpha)N_2^\top & -R\end{bmatrix}\prec 0 \tag{3}$$

holds for $\alpha=\{0,1\}$, where

$$\mathcal{R}^0(\alpha)=\begin{bmatrix}(2-\alpha)R & 0\\ 0 & (1+\alpha)R\end{bmatrix}, \tag{4}$$

then, the following inequality holds

$$\Phi(\alpha)-\Theta(\alpha)\prec 0,\qquad \forall\alpha\in(0,1). \tag{5}$$

Proof. Let us introduce the following positive quadratic term

$$\Pi^\top(\alpha)\begin{bmatrix}\frac{1}{\alpha}R^{-1} & 0\\ 0 & \frac{1}{1-\alpha}R^{-1}\end{bmatrix}\Pi(\alpha)\succeq 0,$$

defined for any α in $(0,1)$, where

$$\Pi(\alpha)=\begin{bmatrix}R & 0\\ 0 & R\end{bmatrix}\Gamma-\begin{bmatrix}\alpha R & 0\\ 0 & (1-\alpha)R\end{bmatrix}\Gamma-\begin{bmatrix}\alpha N_1^\top\\ (1-\alpha)N_2^\top\end{bmatrix}.$$

This inequality indeed holds for any α in $(0,1)$ since the matrix R is assumed to be positive definite and α is positive. Expanding this positive quadratic term leads to

$$\begin{aligned}-\Theta(\alpha)\ \preceq\ &-\Gamma^\top\mathcal{R}^0(\alpha)\Gamma-\mathrm{He}\left(\Gamma^\top\begin{bmatrix}(1-\alpha)N_1^\top\\ \alpha N_2^\top\end{bmatrix}\right)\\ &+\alpha N_1R^{-1}N_1^\top+(1-\alpha)N_2R^{-1}N_2^\top\end{aligned}$$

holds for all α in $(0,1)$, where $\Theta(\alpha)$ is defined in (1). Re-injecting the previous expression of $\Theta(\alpha)$ into the left side of (5), we obtain that, for all $\alpha\in(0,1)$,

$$\begin{aligned}\Phi(\alpha)-\Theta(\alpha)\preceq\ &\Phi(\alpha)-\Gamma^\top\mathcal{R}^0(\alpha)\Gamma-\mathrm{He}\left(\Gamma^\top\begin{bmatrix}(1-\alpha)N_1^\top\\ \alpha N_2^\top\end{bmatrix}\right)\\ &+\alpha N_1R^{-1}N_1^\top+(1-\alpha)N_2R^{-1}N_2^\top.\end{aligned}$$

Since the right-hand-side of the previous inequality is convex with respect to α and is also well defined for $\alpha=\{0,1\}$, the negative definiteness of $\Phi(\alpha)-\Theta(\alpha)$ is guaranteed if, after application of the Schur complement, $\Psi(0)\prec 0$ and $\Psi(1)\prec 0$. Therefore, if the condition (3) is verified for $\alpha=\{0,1\}$, then the inequality $\Phi(\alpha)-\Theta(\alpha)\prec 0$ holds, for all $\alpha\in(0,1)$. $\diamond$

Remark 1. In some cases, instead of considering the matrix $\Theta(\alpha)$, it might be relevant to consider a matrix $\tilde{\Theta}(\alpha)$ given by

$$\tilde{\Theta}(\alpha)=\tilde{\Gamma}^\top\begin{bmatrix}\frac{1}{\alpha}\tilde{R}_1 & 0\\ 0 & \frac{1}{1-\alpha}\tilde{R}_2\end{bmatrix}\tilde{\Gamma},\quad\forall\alpha\in(0,1),$$

where, for given integers n_1, n_2 and m such that $n_1+n_2\leq m$, $\tilde{R}_1$ is in $\mathbb{S}^{n_1}_+$, $\tilde{R}_2$ in $\mathbb{S}^{n_2}_+$, Γ in $\mathbb{R}^{(n_1+n_2)\times m}$, such that $rank(\tilde{\Gamma})=n_1+n_2$. It is noted that the matrix $\tilde{\Theta}(\alpha)$ is defined with two matrices, not necessarily of equal dimension. Lemma 2 can be easily extended to this case without any difficulties, and therefore, will not be presented.