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Brief paper

Robust stabilization of discrete generalized systems[☆]

Sebastian Florin Tudor[a], Cristian Oară[b,*]

[a] School of Business, Stevens Institute of Technology, 1 Castle Point Terrace, Hoboken, NJ 07030, USA
[b] Department of Automatic Control and Systems Engineering, Faculty of Automatic Control and Computers, University Politehnica of Bucharest, Splaiul Independenței 313, Sector 6, RO 060042, Bucharest, Romania

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ABSTRACT

The robust stabilization problem for generalized discrete-time systems described by polynomial or improper transfer function matrices, subject to perturbations acting on the normalized coprime factors, is solved. The maximum achievable stability margin and the robust stabilizing controller are given in terms of realizations and solutions to appropriate Riccati equations.

1. Introduction

Since any mathematical model of a real-life process is inherently an approximation, designing a closed-loop controller that achieves robust stability subject to unstructured model perturbation is an old and central problem in control theory. The novelty of this paper is to consider the robust stabilization problem for the general class of linear discrete-time systems described by transfer matrices which are allowed to be *polynomial* or *improper*. Such systems are alternatively known as *generalized, algebraic-dynamical, singular*, or *descriptor*, and they play an important part in modern control theory, like in the behavioral approach to open and interconnected systems (Willems & Polderman, 1997) which relies on polynomial models, in model predictive control (Rawlings & Mayne, 2009) which involves optimization over future inputs, or in the algebraic analysis and synthesis methods in linear multivariable control (Rosenbrock, 1970; Vardulakis, 1991) which are based on polynomial matrices. From a practical engineering viewpoint, generalized systems provide a great tool for modeling general physical processes as those containing algebraic (non-dynamic) constraints, reversed-time dynamics, or hysteresis (see Blajer, 1992; Dai, 1989; Tolsa & Salichs, 1993). Mass (gas, water, etc.) transportation networks (Offner, Baum, & Kolmbauer, 2016), power and electrical systems (Gunther & Feldmann, 1999; Riaza, 2008), control of robots (Rabier & Rheinboldt, 2000), mechanical systems featuring algebraic constraints (Lind & Schmidt, 2002, chap. 10), and cyber–physical systems under attack (Pasqualetti, Dorfler, & Bullo, 2013) are a couple of examples in which generalized models are key.

The robust stabilization problem has different formulations and has received various solutions according to the class of models and the way in which the uncertainty acts on the nominal plant, additive (Glover, 1986), multiplicative (Stoorvogel, 1996), or on coprime factors (McFarlane & Glover, 1989). The most general and elegant solution for the class of proper linear systems was obtained for the case of additive stable perturbations on the factors in a coprime factorization of the plant, since this family of perturbations is particularly suited for feedback system analysis and contains the other types of uncertainties as particular cases (Vidyasagar, 1985). In McFarlane and Glover (1989), a simple formula for the maximum stability margin together with a characterization of controllers for the normalized coprime factorization of a standard (proper) linear continuous-time system is given in terms of explicit state-space realizations. The discrete-time counterpart of these results for a proper system was obtained in Ionescu, Oară, and Weiss (1999). More recently, robust controllers have been obtained for various classes of models, *e.g.*, nonlinear (Wei & Lin, 2016), stochastic (Kou & Li, 2017), or switched (Wang & Xiang, 2009).

For the class of generalized discrete-time systems subject to additive stable perturbations on the coprime factors we extend the approach in Ionescu et al. (1999) and obtain for the first time numerically-sound realization-based formulas for the stability margin and for the robust stabilizing controller. Our technical tools and derivations are essentially based on centered realizations

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* Corresponding author.
E-mail addresses: studor@stevens.edu (S.F. Tudor), cristian.oara@acse.pub.ro (C. Oară).

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introduced in Rakowski (1992) and used in Rakowski (1994) to parameterize the disturbance decoupling controllers for a proper system. By deriving formulas that have essentially the same simplicity as in the standard proper case (Ionescu et al., 1999; McFarlane & Glover, 1989), we further demonstrate the versatility of centered realizations in approaching robust control problems for this general class of systems. The simplicity of the formulas is due to the use of this particular type of centered realization and could not have been recovered by generalized state-space realizations (Verghese, Levy, & Kailath, 1981; Verghese, Van Dooren, & Kailath, 1979) which are normally involved in the study of this class of singular systems.

The paper is organized as follows. Notation, definitions, and preliminary results are given in Section 2. The robust stabilization problem is formulated and its solution given in Section 3, while all technical proofs are deferred to the Appendix. Section 4 demonstrates our results on a relevant numerical example. Section 5 contains several conclusions.

2. Preliminaries

2.1. General notation and definitions

Denote by $\mathbb{C}, \overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\}, \mathbb{D}, \partial\mathbb{D}, \mathbb{C}^-, \mathbb{C}^+, j\mathbb{R}, j\overline{\mathbb{R}} := j\mathbb{R} \cup \{\infty\}$ the complex plane, the extended complex plane, the open unit disk, the unit circle, the open left-half plane, the open right-half plane, the imaginary axis, and the extended imaginary axis, respectively.

For a matrix $A \in \mathbb{C}^{p\times m}$, A^* is its conjugate transpose and $\sigma_{\max}(A)$ is its maximum singular value. If A is square, $\rho(A)$ denotes its spectral radius. The $p \times m$ matrix polynomial $A - \lambda E$ is called a (matrix) pencil, where λ is a variable in $\mathbb{C}$. The pencil is called *regular* if it is square and $\det(A-\lambda E) \not\equiv 0$. $\Lambda(A-\lambda E)$ is the union of generalized eigenvalues (finite and infinite, multiplicity counting) of the regular pencil $A - \lambda E$ (see for example Gantmacher, 1960).

Throughout the paper a linear dynamical system, with m inputs and p outputs, is formally described by its *transfer function matrix* (TFM)

$$\mathbf{H}(\lambda) = \left[\frac{a_{ij}(\lambda)}{b_{ij}(\lambda)}\right]_{\substack{i=\overline{1,p}\\ j=\overline{1,m}}}, \tag{1}$$

with $a_{ij}(\lambda)$ and $b_{ij}(\lambda)$ scalar polynomials with coefficients in $\mathbb{C}$. The focus of this paper is on *generalized systems* whose TFMs may be *improper*, *i.e.*, $\deg a_{ij} > \deg b_{ij}$, or *polynomial*, *i.e.*, $b_{ij} \equiv 1$, for some i, j. Denote the set of all $p \times m$ complex TFMs by $\mathbb{C}^{p\times m}(\lambda)$.

Let $\Omega \subset \mathbb{C}$ denote either $\mathbb{D}$ or $\mathbb{C}^-$, and $\partial\Omega$ its boundary, *i.e.*, either $\partial\mathbb{D}$ or $j\overline{\mathbb{R}}$. We say that a system $\mathbf{H}(\lambda)$ is *Ω-stable* provided all its poles are in Ω. We shall denote the ring of all Ω-stable TFMs with $\mathcal{RH}_\infty(\Omega)$. Let $\mathcal{RL}_\infty(\partial\Omega)(\supset \mathcal{RH}_\infty(\Omega))$ be the Banach space of complex $p \times m$ TFM bounded on $\partial\Omega$, having the $\mathcal{H}_\infty$ norm defined as $\|\mathbf{H}\|_\infty^{\partial\Omega} := \sup_{\omega\in\partial\Omega}\sigma_{\max}\Big(\mathbf{H}(\omega)\Big)$ (for more details see Section 4.3 in Zhou, Doyle, and Glover, 1996).

2.2. Realizations for generalized systems

An alternative representation of linear dynamical systems, prone to numerically-sound computations, is through realizations. Precisely as for the solution in the standard case (Ionescu et al., 1999; McFarlane & Glover, 1989), we use appropriate realizations as a main vehicle to obtain reliable analytical formulas. Generalized systems are usually represented by so-called *generalized state-space realizations* (see Dai, 1989; Verghese et al., 1981)

$$\mathbf{H}(\lambda) = C(\lambda E - A)^{-1}B + D =: \left[\begin{array}{c|c} A-\lambda E & B \\ \hline C & D \end{array}\right], \tag{2}$$

where $A - \lambda E$ is a regular $n \times n$ pencil and all the intervening constant matrices A, E, B, C, D have complex elements and appropriate dimensions. Generalized state-space realizations can cope with the presence of poles at ∞ and are an extension of standard realizations. Although (2) is suited to represent any TFM model it has a couple of drawbacks for the problems under investigation: if ∞ is a pole of $\mathbf{H}(\lambda)$ then the order n of the realization (2) is strictly greater than the McMillan degree of $\mathbf{H}(\lambda)$, minimality of a realization is not equivalent to controllability + observability, two realizations having minimal order are not necessarily related by an equivalence transformation, and starting from an arbitrary realization one cannot in general obtain a minimal one by unitary transformations only (Dai, 1989; Van Doreen, 1981b; Verghese et al., 1981, 1979).

To circumvent these shortcomings, we will work with a slightly more general type of realization, called *centered*, introduced in Rakowski (1992). To define a centered realization fix first a $\lambda_0 \in \overline{\mathbb{C}}$, and further α, β, such that if $\lambda_0 = \infty$ then $\alpha = 1$ and $\beta = 0$, and if $\lambda_0 \in \mathbb{C}$ then $\frac{\alpha}{\beta} = \lambda_0$. A *realization centered at* λ_0 is a representation

$$\mathbf{H}(\lambda) = D + C(\lambda E - A)^{-1}B(\alpha - \beta\lambda) =: \left[\begin{array}{c|c} A-\lambda E & B \\ \hline C & D \end{array}\right]_{\lambda_0}, \tag{3}$$

where $A - \lambda E$ is a regular pencil, $A, E \in \mathbb{C}^{n\times n}$, $B \in \mathbb{C}^{n\times m}$, $C \in \mathbb{C}^{p\times n}$, and $D \in \mathbb{C}^{p\times m}$. In particular, if $\lambda_0 = \infty$ we drop the index λ_0 from (3) and get precisely the notation and representation in (2). Therefore, realizations (2) are simply realizations centered at $\lambda_0 = \infty$. The positive integer n is called the *order* of the realization (3). For an improper or polynomial TFM the matrix E is always singular, with $\operatorname{rank} E < n$. The realization (3) (or the pair $(A - \lambda E, B)$) is called *controllable at* $\lambda \in \mathbb{C}$ if $\operatorname{rank}\,[A - \lambda E\ B] = n$, and is called *controllable at* ∞ if $\operatorname{rank}\,[E\ B] = n$. Analogously, (3) is called *observable* (or the pair $(C, A - \lambda E)$ is *observable*) at a certain $\lambda \in \overline{\mathbb{C}}$ provided the pair $(A^* - \lambda E^*, C^*)$ is controllable at λ. A realization (or a pair) is called *controllable* (*observable*) provided it is controllable (observable) $\forall\lambda \in \overline{\mathbb{C}}$. A realization (3) (or the pair $(A - \lambda E, B)$) is called *Ω-stabilizable* if it is controllable for all $\lambda \in \overline{\mathbb{C}} \setminus \Omega$. Analogously, the pair $(C, A - \lambda E)$ is called *Ω-detectable* if $(A^* - \lambda E^*, C^*)$ is Ω-stabilizable. The realization is called *minimal* if its order is as small as possible among all realizations centered at the given λ_0.

The key features of centered realizations are revealed by choosing λ_0 different from any pole of $\mathbf{H}(\lambda)$ — a choice in force henceforth in the paper. In this case, one recovers all nice properties of standard realizations (Rakowski, 1992), eliminating therefore all the aforementioned drawbacks of generalized state-space realizations of type (2): the order of a minimal realization (3) equals the McMillan degree of $\mathbf{H}(\lambda)$; minimality of a realization (3) is equivalent to controllability plus observability; $D = \mathbf{H}(\lambda_0)$; any two minimal realizations

$$\mathbf{H}(\lambda) = \left[\begin{array}{c|c} A_1-\lambda E_1 & B_1 \\ \hline C_1 & D_1 \end{array}\right]_{\lambda_0} = \left[\begin{array}{c|c} A_2-\lambda E_2 & B_2 \\ \hline C_2 & D_2 \end{array}\right]_{\lambda_0}$$

are related by an *equivalence transformation* defined by invertible matrices Q, Z, such that

$$A_2 - \lambda E_2 = Q(A_1 - \lambda E_1)Z, \quad B_2 = QB_1, \quad C_2 = C_1Z; \tag{4}$$

starting from any realization (3) one can always extract a minimal one by using unitary transformations only.

An additional feature of centered realizations is the easiness in obtaining them, similar to the standard case, see (Rakowski, 1992) for a method to get directly a realization (3) starting from the TFM (1), and Section 5 in Oară and Sabău (2009) for a procedure to switch back and forth between realizations (2) and (3).