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Brief paper

The analysis of nonlinear systems in the frequency domain using Nonlinear Output Frequency Response Functions☆

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ABSTRACT

The Nonlinear Output Frequency Response Functions (NOFRFs) are a concept which provides a new extension of the well-known concept of the Frequency Response Function (FRF) of linear systems to the nonlinear case. The present study introduces a NOFRFs based approach for the analysis of nonlinear systems in the frequency domain. It is well known that a nonlinear system can, under rather general conditions, be represented by a polynomial type Nonlinear Auto Regressive with eXogenous input (NARX) model. From the NARX model of a nonlinear system under study, the NOFRFs based approach for the frequency analysis of nonlinear systems involves solving a set of linear difference equations known as the Associated Linear Equations (ALEs) to determine the system nonlinear output responses and then the NOFRFs of the system up to an arbitrary order of nonlinearity of interests. The results enable a representation of the frequency domain characteristics of nonlinear systems by means of a series of Bode diagram like plots that can be used for nonlinear system frequency analyses for various purposes including, for example, condition monitoring, fault diagnosis, and nonlinear modal analysis.

1. Introduction

The frequency domain approach of linear systems is the very basis of control, signal processing, and communication and has been applied in almost all science and engineering areas. The key concept associated with the linear system frequency analysis is the Frequency Response Function (FRF), which is the foundation of Bode diagrams, Nyquist stability criterion, modal analysis, filter designs, among other well-known and well-established theories and methods.

The direct extension of the FRF concept to the nonlinear case is known as the Generalized Frequency Response Functions (GFRFs) (George, 1959), which were proposed under the assumption that the output of the nonlinear systems under study can be described by a convergent Volterra series (Boyd & Chua, 1985). The difficulties with the practical application of the GFRFs are that the GFRFs can only be graphically studied up to the second order (Yue, Billings, & Lang, 2005). This implies that the well-established Bode or Nyquist diagram based frequency domain analysis cannot be generally extended to the nonlinear case. Therefore, although some specific applications can be found in literatures such as, e.g., in image processing (Ramponi, 1986), channel equalization (Karam & Sari, 1989) and fault detection (Tang, Liao, Cao, & Xie, 2010), a systematic approach for the analysis of nonlinear systems in the frequency domain that can be generally applied in practice still does not exist.

It is worth mentioning that describing functions (Khalil, 2002) are a traditional frequency domain analysis approach to nonlinear systems which only involve a one dimensional function of frequency and have been used in practical nonlinear system control problems. However, describing functions are defined for specific nonlinear components and can only be applied in the context of simple control systems with an *a priori* given structure.

Nonlinear FRF and associated nonlinear Bode plots (Pavlov, van de Wouw, & Nijmeijer, 2007; Rijlaarsdam, Nuij, Schoukens, & Steinbuch, 2017) were introduced based on the exact evaluation of the bound on the output response of nonlinear systems under a harmonic excitation. These are the concepts of the nature and properties similar to that of the describing functions.

In order to resolve these difficulties, researchers have made considerable efforts to develop new concepts that can capture the system essential features while keeping problem dimensionality low. Examples of such approaches are the best linear approximation (Schoukens, Nemeth, Crama, Rolain, & Pintelon, 2003), the High Order Sinusoidal Input Describing Functions (HOSIDF) (Nuij, Bosgra, & Steinbuch, 2006) and the Associated Frequency

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Response Functions (AFRFs) (Feijoo, Worden, & Stanway, 2004). These approaches have also been applied to solve many engineering problems (Feijoo, Worden, & Stanway, 2006; Rijlaarsdam, Nuij, Schoukens, & Steinbuch, 2012; Rijlaarsdam, Setiadi, Nuij, Schoukens, & Steinbuch, 2013). However, these approaches have many limitations. For instance, HOSIDF can only deal with sinusoidal inputs and require complex computations that must be repeated for each frequency of interest while AFRFs can be evaluated only when the differential equation model of the system under study is available. In addition, these approaches have only been studied for simple and particular cases. It is difficult to assess the efficiency of these approaches in situations where systems are described by more general nonlinear models.

The concept of Nonlinear Output Frequency Response Functions (Lang & Billings, 2005) – NOFRFs – is a new extension of the FRF to the nonlinear case. One of its most attractive feature is its one-dimensional nature, which has many advantages, as has been demonstrated by a wide range of studies (Lang & Peng, 2008; Peng, Lang, & Billings, 2007). However, current applications of the NOFRFs use a Least Squares (LS) based method to evaluate the NOFRFs (Lang & Billings, 2005). This requires an appropriate selection of the maximum order of the system nonlinearity, which is sometimes difficult and may suffer from numerical issues. In addition, the method requires the system response data from several simulation or experimental tests, which may not be convenient for implementation.

The present study is motivated by the need of addressing these problems. A systematic NOFRF-based approach for the nonlinear system frequency analysis is developed based on a polynomial type Nonlinear Auto Regressive with eXogenous input (NARX) model, which can either be obtained by discretizing the system's nonlinear differential equation model or determined by a data driven system identification method (Billings, 2013). The work involves the derivation of an algorithm which solves a set of linear difference equations to determine the nonlinear output responses and then the NOFRFs of a nonlinear system up to an arbitrary order of interest.

The results enable a representation of the frequency domain characteristics of nonlinear systems by means of a series of Bode diagram like plots that can be used for nonlinear system frequency analyses for various purposes including, for example, condition monitoring, fault diagnosis, and nonlinear modal analysis (Xia, Ni, & Sang, 2017; Zhang, Lang, & Zhu, 2016). The application of the proposed new analysis to the detection and quantification of cracks in a beam structure is finally demonstrated in a case study.

2. The NOFRFs based approach for the analysis of nonlinear systems in the frequency domain

2.1. Nonlinear Output Frequency Response Functions (NOFRFs)

Let $y(k)$ and $u(k)$ respectively denote the output and input of a discrete time fading memory system (Boyd & Chua, 1985) with a zero equilibrium, and k represent the discrete time. The system output response around the origin can be described by the Volterra series:

$$y(k) = \sum_{n=1}^{+\infty} y_n(k) = \sum_{n=1}^{+\infty} \sum_{\mathbb{Z}^n} h_n(\boldsymbol{\tau}_n) \prod_{i=1}^{n} u(k-\tau_i) \quad (1)$$

where $y_n(k)$ denotes a degree-n polynomial functional of $u(k)$, $h_n(\boldsymbol{\tau}_n) = h_n(\tau_1, \ldots, \tau_n)$ is the degree-n kernel.

Functionals can be described in the frequency domain using integral transforms such as the Z transform or the normalized Discrete-Time Fourier Transform (DTFT). For example, the normalized DTFT of $y_n(k)$ can be described as (Lang & Billings, 1996)

$$Y_n(\mathrm{j}\omega) = \frac{1}{\sqrt{n}(2\pi)^{n-1}} \int H_n(\mathrm{j}\boldsymbol{\omega}_n) \prod_{i=1}^{n} U(\mathrm{j}\omega_i)\, \mathrm{d}\sigma_{n,\omega} \quad (2)$$

where the integration is carried out over the hyperplane $\omega_1 + \cdots + \omega_n = \omega$ with $-\pi \le \omega/f_s \le \pi$, where f_s is the sampling frequency.

The function $H_n(\mathrm{j}\boldsymbol{\omega}_n) = H_n(\omega_1, \ldots, \omega_n)$ is the nth order GFRF defined as the nth order normalized DTFT of $h_n(\boldsymbol{\tau}_n)$

$$H_n(\mathrm{j}\boldsymbol{\omega}_n) = \Delta t \sum_{\mathbb{Z}^n} h_n(\boldsymbol{\tau}_n) \prod_{i=1}^{n} e^{-\mathrm{j}\omega_i\tau_i\Delta t} \quad (3)$$

and $U(\mathrm{j}\omega)$ is the normalized DTFT of $u(k)$.

Definition 1. Let $u(k)$ be the sequence of a finite energy signal. In the discrete time domain, the nth order generalized spectrum of $u(k)$ is defined as (Lang & Billings, 2005):

$$U_n(\mathrm{j}\omega) = DF\left\{u^n(k)\right\}\Delta t = \frac{1}{\sqrt{n}(2\pi)^{n-1}} \int \prod_{i=1}^{n} U(\mathrm{j}\omega_i)\, \mathrm{d}\sigma_{n,\omega} \quad (4)$$

where DF denotes the DTFT.

Definition 2. The nth order NOFRF is defined as (Lang & Billings, 2005):

$$G_n(\mathrm{j}\omega) = \frac{Y_n(\mathrm{j}\omega)}{U_n(\mathrm{j}\omega)};\ \omega \in \Omega \subseteq [-\pi f_s, \pi f_s] \quad (5)$$

where Ω is the frequency support of $|U_n(\mathrm{j}\omega)|$, which can be determined using the results about the output frequencies of nonlinear systems (Lang & Billings, 1996).

The NOFRFs as defined in (5) have the following attractive properties.

Property 1 ((*Lang & Billings, 2005*)). *Let K be a non-zero constant and $G_n(\mathrm{j}\omega)$ the nth order NOFRF computed for $U(\mathrm{j}\omega)$. Then, the NOFRF computed for $KU(\mathrm{j}\omega)$ are also $G_n(\mathrm{j}\omega)$.*

Property 2 ((*Lang & Billings, 2005*)). *The frequency support of $G_n(\mathrm{j}\omega)$, $Y_n(\mathrm{j}\omega)$ and $U_n(\mathrm{j}\omega)$, i.e., the frequency range where these functions of frequency are well defined, are the same.*

2.2. The NOFRFs based approach for the analysis of nonlinear systems in the frequency domain

It is obvious that the NOFRFs are an extension of the FRF to the nonlinear case, as when $n = 1$, $G_n(\mathrm{j}\omega) = G_1(\mathrm{j}\omega)$ reduces to the FRF of a linear system.

The NOFRFs of higher orders are generally dependent on the system input (Lang & Billings, 2005). However, different systems have different NOFRFs when probed by the same input. Consequently, the NOFRFs evaluated under the same input can be exploited to reveal the differences between systems in the frequency domain. This is the fundamental idea of the NOFRFs based system frequency analysis.

Based on these ideas, a general approach for the analysis of nonlinear systems in the frequency domain using the NOFRFs can be proposed as follows.

(i) Find an NARX model of the nonlinear system.
(ii) Determine the NOFRFs of the system from the NARX model under a probing input dependent on the specific application.
(iii) Analyze the system in the frequency domain from the determined NOFRFs for the specific application related objective.