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Signed bounded confidence models for opinion dynamics ☆

Claudio Altafini [a,*], Francesca Ceragioli [b]

[a] Division of Automatic Control, Department of Electrical Engineering, Linköping University, SE-58183, Linköping, Sweden
[b] Dip. di Scienze Mathematiche, Politecnico di Torino, Corso Duca degli Abruzzi, 24, 10129, Italy

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ABSTRACT

The aim of this paper is to modify continuous-time bounded confidence opinion dynamics models so that "changes of opinion" (intended as changes of the sign of the initial states) are never induced during the evolution. Such sign invariance can be achieved by letting opinions of different sign localized near the origin interact negatively, or neglect each other, or even repel each other. In all cases, it is possible to obtain sign-preserving bounded confidence models with state-dependent connectivity and with a clustering behavior similar to that of a standard bounded confidence model.

1. Introduction

A bounded confidence model is a model of consensus-like opinion dynamics in which the agents interact with each other only when their opinions are close enough. Such a class of models usually goes under the name of Hegselmann–Krause models (Hegselmann & Krause, 2002) and has the peculiarity of expressing confidence as a function of the distance between the agents states. As a consequence, the graph that describes the interactions between the agents is itself state-dependent and varying in time. The emerging behavior of such a model is that the agents tend to form clusters, and a consensus value is achieved among the agents participating to a cluster. In the control literature, various aspects of such models have been studied: discrete-time (Blondel, Hendrickx, & Tsitsiklis, 2009b; Etesami & Basar, 2015; Mohajer & Touri, 2013), continuous-time (Blondel, Hendrickx, & Tsitsiklis, 2010; Motsch & Tadmor, 2014; Tay Stamoulas & Rathinam, 2015), and stochastic (Como & Fagnani, 2011) dynamics, convergence time (Coulson, Steeves, Gharesifard, & Touri, 2015; Mohajer & Touri, 2013), behavior of a continuum of agents (Hendrickx & Olshevsky, 2016), existence of interaction rules that allow to preserve the connectivity (Yang, Dimarogonas, & Hu, 2014), presence of stubborn agents (Frasca, Ravazzi, Tempo, & Ishii, 2013) etc. See Frasca, Ishii, Ravazzi, and Tempo (2015), Friedkin (2015) and Lorentz (2007) for an overview. In continuous time, if the confidence range is delimited by a sharp threshold, then the right hand side of the resulting ODEs is discontinuous. Existence and uniqueness analysis of the corresponding solutions has been carried out in Blondel et al. (2010) and Ceragioli and Frasca (2012). In Ceragioli and Frasca (2012) approximations of the discontinuous dynamics are suggested.

In the social sciences literature, many models have been proposed to represent opinion dynamics and interpersonal influences in a social network of individuals (Freeman, 2004; Harary, Cartwright, & Norman, 1965; Scott, 2012; Wasserman & Faust, 1994). A system-theoretical overview of some of these models, like for instance the French–DeGroot model (consensus-like behavior, without any distance-dependent bound, DeGroot, 1974; French, 1956) or its Friedkin–Johnsen generalization (mixture of consensus and stubbornness, Friedkin & Johnsen, 1999) is given in Proskurnikov and Tempo (2017), where many more pointers to relevant papers are provided. Alongside a vast theoretical research, the field of experimental social psychology has produced a number of empirical studies (mainly involving small social groups) meant to validate such social opinion change models. There is a wide consensus in this literature that the only experimental feature that can be consistently documented in this context is that opinions are constrained to the convex hull of the initial conditions, but that the sensitivity of an individual to influences is a subjective parameter, varying widely across a community of individuals (Friedkin & Johnsen, 2011). Evidence of a threshold on the confidence range does not seem to be documented in this literature. In spite of the lack of empirical validation, from a dynamical point of view the behavior of a bounded confidence model is interesting as a

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* Corresponding author.
E-mail addresses: claudio.altafini@liu.se (C. Altafini), d003432@polito.it (F. Ceragioli).

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mechanism for the formation of clusters of agents, according only to the initial conditions on the ODEs. It is in view of its rich dynamical behavior and of the nontrivial mathematics induced by state-dependence of the interaction graph that we have decided to adopt it in this paper.

For the bounded confidence model, there is a special situation in which confidence between the agents may be lost even if the opinions are in proximity, and it is when the signs of the opinions are different. It is intuitively clear that "changing opinion", intended as changing sign of an agent's opinion, is a fairly drastic process, a "mental barrier" not so likely to be trespassed in real scenarios. Currently available bounded confidence models only consider the value of the opinions relative to each other, and do not distinguish between the case of all opinions having the same sign or less, i.e., the opinions can freely cross zero while converging to a local consensus value. In other words, the bounded confidence models are translationally invariant.

The aim of this paper is to propose models of bounded confidence in which translational invariance is replaced by preservation of the signs of the original opinions. Several possible ways to implement this principle exist, and in fact in this paper we propose 3 different models. Their common basis is that opinions having the same sign attract each other, while opinions of different sign can lead to negative interactions, indifference or even repulsion. Consequently, the dynamics among opinions of different signs can be constructed according to different rules. The simplest possibility is to make use of the notion of bipartite consensus introduced in Altafini (2013a). Under certain conditions on the graph of the signed interactions, the agents split into two groups converging to a consensus value which is equal in modulus but opposite in sign. The graphical condition that needs to be fulfilled, called structural balance (Altafini, 2013a), is naturally satisfied when initial conditions that have the same sign are associated to positive edges ("friends") and those having opposite signs to negative edges ("enemies"). The sign function used in the model to make this distinction implies that even when no bound on the confidence is present, the connectivity is state-dependent: the graph describing interactions among agents depends on the initial conditions. In spite of a discontinuous right-hand side, this model almost always has unique solutions. Only when one or more of the initial opinions are 0, then multiple Carathéodory solutions arise. When a bound is added on the confidence range, then the negative interactions among agents are only localized around the origin and do not affect the asymptotic behavior of opinions far from 0. Even with the negative interactions around the origin, almost all initial conditions are however *proper* (i.e., lead to a unique solution which can be prolonged to $+\infty$ without incurring in accumulation of nondifferentiability points). The overall behavior of the model is still to create clusters of agents achieving a common consensus value within each cluster while in addition preserving the sign of all initial conditions.

The behavior in terms of existence and uniqueness of the solutions, as well as in terms of the asymptotic clustering, is similar if in the model agents having opposite opinions simply ignore each other. Also in this case, in fact, a (Heaviside) sign function must be introduced in order to suppress the contribution of nearby opinions of different sign in the bounded confidence dynamics. The discontinuities of the sign function may give rise to multiple Carathéodory solutions. However, almost all initial conditions are still proper and lead to the formation of clusters of opinions.

Finally, when sign discordance is modeled as a repulsion term, the combination of sign preservation and bounded confidence can give rise to more complex behaviors in which solutions á la Carathéodory are not guaranteed to exist. In the third model we give, the repulsion dynamics may lead to discontinuities which are attractive, meaning that the opinion may stay on the discontinuity value while forming clusters. As in the previous models, the resulting solutions (now of Krasovskii type) have the property of preserving the sign of the original opinions, i.e., no agent has to change its mind during the time evolution of the system.

A preliminary version of this material was presented at the 2016 European Control Conference, see Ceragioli et al. (2016). This conference paper deals only with the first of the three models discussed in the current manuscript. The other two variants are novel material presented here for the first time.

The rest of this paper is organized as follows. After recalling the necessary background material in Section 2, in Section 3 we introduce the three models of signed bounded confidence and describe their dynamical behavior in what is the main theorem of the paper. To illustrate their differences, in Section 4 the three models are studied in absence of any confidence bound. Finally Section 5 contains the proof of the main theorem and a series of examples.

2. Background material

2.1. Linear algebraic notions

A matrix $A \in \mathbb{R}^{n\times n}$ is said Hurwitz stable if all its eigenvalues $\lambda_i(A), i = 1, \dots, n$, have $\mathrm{Re}[\lambda_i(A)] < 0$. It is said marginally stable if $\mathrm{Re}[\lambda_i(A)] \leqslant 0, i = 1, \dots, n$, and $\lambda_i(A)$ such that $\mathrm{Re}[\lambda_i(A)] = 0$ have an associated Jordan block of order one. A is said irreducible if there does not exist a permutation matrix Π such that $\Pi^T A \Pi$ is block triangular. The matrices A considered in this paper will always be symmetric: $A = A^T$. A matrix A is said *diagonally dominant* if

$$|A_{ii}| \geqslant \sum_{j\neq i} |A_{ij}|, \quad i = 1, \dots, n. \tag{1}$$

It is said *strictly diagonally dominant* when all inequalities of (1) are strict, and *weakly diagonally dominant* when at least one (but not all) of the inequalities (1) is strict. A is said *diagonally equipotent* (Altafini, 2013b) if

$$|A_{ii}| = \sum_{j\neq i} |A_{ij}|, \quad i = 1, \dots, n.$$

2.2. Signed graphs

Given a matrix $A = A^T \in \mathbb{R}_+^{n\times n}$, consider the undirected graph $\Gamma(A)$ of A: $\Gamma(A) = \{\mathcal{V}, A\}$ where $\mathcal{V} = \{1, \dots, n\}$ is the set of n nodes and A is its weighted adjacency matrix. Self weights are excluded from A: $A_{ii} = 0$. $\Gamma(A)$ is connected if there exists a path between each pair of nodes in $\mathcal{V}$. It is fully connected if $A_{ij} \neq 0$ $\forall\, i \neq j$. An adjacency matrix that can assume both positive and negative values is denoted A_s and its associated signed graph $\Gamma(A_s)$. An undirected signed graph $\Gamma(A_s)$ is said *structurally balanced* if all its cycles are positive (i.e., they have an even number of negative edges). $\Gamma(A_s)$ is structurally balanced if and only if there exists a vector $s = [s_1 \cdots s_n]$, $s_i = \pm 1$, such that the matrix $A = S A_s S$ is nonnegative definite, where $S = \mathrm{diag}(s)$ is the diagonal matrix having the entries of s on the diagonal.

2.3. Bipartite consensus

Given a matrix A, $A_{ij} \geq 0$ for $i \neq j$, the *(standard) Laplacian* associated with A is the matrix L of elements

$$L_{ij} = \begin{cases} -A_{ij} & \text{if } i \neq j \\ \sum_{k\neq i} A_{ik} & \text{if } i = j. \end{cases}$$

The linear system

$$\dot{x} = -Lx \tag{2}$$