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Reliable non-linear state estimation involving time uncertainties☆

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ABSTRACT

This paper presents a new approach to bounded-error state estimation involving time uncertainties. For a given bounded observation of a continuous-time non-linear system, it is assumed that neither the values of the observed data nor their acquisition instants are known exactly. For systems described by state-space equations, we prove theoretically and demonstrate by simulations that the proposed constraint propagation approach enables the computation of bounding sets for the systems' state vectors that are consistent with the uncertain measurements. The bounding property of the method is guaranteed even if the system is strongly non-linear. Compared with other existing constraint propagation approaches, the originality of the method stems from our definition and use of bounding tubes which enable to enclose the set of all feasible trajectories inside sets. This method makes it possible to build specific operators for the propagation of time uncertainties through the whole trajectory. The efficiency of the approach is illustrated on two examples: the dynamic localization of a mobile robot and the correction of a drifting clock.

1. Introduction

This paper presents a novel method for the state estimation of a dynamical system of the form:

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{f}\big(\mathbf{x}(t), \mathbf{u}(t)\big), & \text{(a)} \\ z_i = g\big(\mathbf{x}(t_i)\big), & \text{(b)} \end{cases} \tag{1}$$

where $\mathbf{x}(t) \in \mathbb{R}^n$ is the state vector representing the system at time t and $\mathbf{f} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ a non-linear function depicting the evolution of the system based on input vectors $\mathbf{u}(t) \in \mathbb{R}^m$. The observation function $g : \mathbb{R}^n \to \mathbb{R}$ is assumed to be scalar, without loss of generality as the methods are readily scalable to the vector case. The $t_i, i \in \mathbb{N}$, are measurement times and the z_i are the related outputs.

In a bounded-error approach to the state estimation problem, we can assume the function $\mathbf{f}$ and the measurements z_i are not known exactly. Instead, we shall consider that $\mathbf{f}$ is represented by a set-valued function $[\mathbf{f}]$ and that measurements z_i all belong to some known intervals denoted by $[z_i]$. When the t_i are exactly known, interval analysis (Moore, 1966) combined with constraint propagation (Bessiere, 2006; Gning & Bonnifait, 2006) is able to efficiently solve the state estimation problem (Jaulin, Kieffer, Braems, & Walter, 2001; Milanese & Vicino, 1991; Raïssi, Ramdani, & Candau, 2004). More precisely, without any prior knowledge on the state, an interval calculus allows to compute for each t a set enclosing all feasible state vectors.

This paper deals with uncertain measurement times: the t_i are only known to belong to some interval $[t_i]$. In this context, neither the value of the output z_i nor the acquisition date t_i are known exactly. Hence, the problem becomes much more complex as the uncertainties related to the t_i are difficult to propagate through the differential equation. Some attempts of using interval analysis have been proposed in Bethencourt and Jaulin (2013) and Le Bars, Sliwka, Jaulin, and Reynet (2012), but the corresponding observers cannot be considered as guaranteed. Other works, often referred as Out Of Sequence Measurement (OOSM) (Choi, Choi, Park, & Chung, 2009), state problems of time delay uncertainties, which can be somehow related to our problem. However, the considered time uncertainties are tight, of the same order of magnitude as computational time step, and treated by means of covariance matrices which do not provide guaranteed results. In contrast, this paper proposes a reliable computational tool set to deal with strong temporal uncertainty constraints in systems involving differential equations and non-linear functions.

Motivating application. Some practical problems can be formulated to deal with time uncertainties. As an illustration, let us

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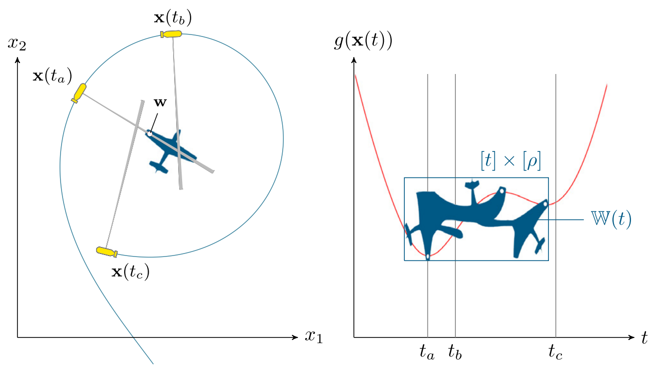

Fig. 1. A robot $\mathcal{R}$ perceiving a plane wreck by using a side scan sonar. The observation function $g(\mathbf{x})$ represents the distance between $\mathcal{R}$ and a point of interest $\mathbf{w}$ on the plane, pictured by a white dot and seen at times $t_1 = t_a, t_2, t_3$. The sonar image $\mathbb{W}(t)$ is overlaid on the graph. Although $\mathbf{w}$ has been seen three times, the t_i remain uncertain but known to belong to $[t]$. Some other robot states are illustrated at times t_a, t_b, t_c.

consider an underwater robot $\mathcal{R}$ performing an exploration task using a side-scan sonar. Assume that a localization of the robot is based on the perception of a wreck for which the highest point $\mathbf{w}$ is precisely geolocalized. As pictured in Fig. 1, the wreck image $\mathbb{W}(t)$ obtained by the sonar may be distorted, stretched and would be highly noisy in practice, depending on the robot navigation (Le Bars et al., 2012). It is a difficult problem for image processing algorithms to detect the highest point $\mathbf{w}$ in $\mathbb{W}$ to be used as reference for localization. However, the problem can be dealt with in a temporal way, based on the time interval $[t]$ during which the robot has seen the wreck. This observation is related to a strong temporal uncertainty: up to several seconds or minutes. Then the state estimation amounts to a range-only problem for which $\exists t \in [t], \exists \rho \in [\rho] \mid \rho = g(\mathbf{x}(t))$, with $g : \mathbb{R}^n \to \mathbb{R}$ the distance function between $\mathcal{R}$ and the known point $\mathbf{w}$.

This example shows how a classic robotic application can be related to strong time uncertainties. The current paper is a first step towards new state estimation approaches that will focus on both the time and the state spaces. It proposes a theoretical basis to deal with the former in the most generic way and is illustrated by reproducible examples in order to highlight the interest and simplicity of the method and encourage further comparisons.

This paper is organized as follows. Section 2 gives an overview of constraint propagations related to sets of trajectories, introducing the concept of Constraint Networks, tubes and contractors. These tools will then be extended to the time uncertainty constraint this paper is dealing with. The approach, theoretically detailed in Section 3, will be illustrated through two robotics examples. The first one, Section 4, involves a mobile robot to be localized while evolving amongst beacons emitting uncertain range-only signals. The second one, detailed in Section 5, provides an original method to correct a drifting clock using ephemeris measurements. Sections 6 and 7 conclude the paper and present the numerical libraries used during this work.

2. Constraint propagation over trajectories

Section 2.1 recalls the principle of constraint propagation (Bessiere, 2006; Van Hentenryck, Michel, & Benhamou, 1998) that will be used later to formalize problems concerning dynamical systems. To this end, a *tube* can be used to enclose a feasible solution set: an envelope of trajectories compliant with the selected constraints. The notion of tube is recalled in Section 2.2 with related properties.

2.1. Constraint networks

In a numerical context, problems of control, state estimation and robotics can be described as *Constraint Networks* (CNs), in which variables must satisfy a set of rules or facts, called *constraints*, over domains defining a range of feasible values. Links between the constraints define a network (Mackworth, 1977) involving variables $\{x_1, \ldots, x_n\}$, constraints $\{\mathcal{L}_1, \ldots, \mathcal{L}_m\}$ and domains $\{\mathbb{X}_1, \ldots, \mathbb{X}_n\}$ containing the x_i's. The variables x_i can be symbols, real numbers (Araya, Trombettoni, & Neveu, 2012) or vectors of $\mathbb{R}^n$. The constraints can be non-linear equations between the variables, such as $x_3 = \cos\big(x_1 + \exp(x_2)\big)$. Domains can be intervals, boxes (Jaulin, Kieffer, Didrit, & Walter, 2001), or polytopes (Combastel, 2005).

Contractors. A constraint $\mathcal{L}$ can be applied on a box $[\mathbf{x}] \in \mathbb{IR}^n$ with the help of a contractor $\mathcal{C}$. The box $[\mathbf{x}]$, also called *interval-vector*, is a closed and connected subset of $\mathbb{R}^n$ and belongs to the set of n-dimensional boxes denoted by $\mathbb{IR}^n$. Formally, a contractor $\mathcal{C}$ associated to the constraint $\mathcal{L}$ is an operator $\mathbb{IR}^n \to \mathbb{IR}^n$ that returns a box $\mathcal{C}([\mathbf{x}]) \subseteq [\mathbf{x}]$ without removing any vector consistent with $\mathcal{L}$. Constructing a store of contractors such as $\mathcal{C}_+, \mathcal{C}_{\sin}, \mathcal{C}_{\exp}$ associated to primitive equations such as $z = x + y$, $y = \sin(x)$, $y = \exp(x)$ has been the subject of much work (Chabert & Jaulin, 2009; Desrochers & Jaulin, 2016; Jaulin, Kieffer, Didrit et al., 2001).

Decomposition. Problems involving complex equations can be broken down into a set of primitive equations. Here, *primitive* means that the constraint cannot be decomposed anymore and that the related operator is available in a collection of contractors, thus allowing to deal with a wide range of problems. For instance, the non-linear equation $x_3 = \cos\big(x_1 + \exp(x_2)\big)$ can be decomposed into:

$$\begin{cases} a = \exp(x_2), \\ b = x_1 + a, \\ x_3 = \cos(b). \end{cases} \tag{2}$$

Combining primitive contractors leads to a complex contractor that still provides reliable results (Chabert & Jaulin, 2009).

Propagation. When working with finite domains, a propagation technique can be used to solve a problem. The process is run up to a fixed point when domains $\mathbb{X}_i$ cannot be reduced anymore.

Our goal is to consider trajectories as variables and to implement contractors to reduce their domains given a constraint that can be algebraic or differential. This will be done by using tubes as domains for these variables.

2.2. Tubes: envelopes of feasible trajectories

In this paper, the notation $(\cdot)$ is used in order to clearly distinguish a whole trajectory $\mathbf{x}(\cdot) : \mathbb{R} \to \mathbb{R}^n$ from a local evaluation: $\mathbf{x}(t) \in \mathbb{R}^n$.

Definition. A tube is defined (Filippova, Kurzhanski, Sugimoto, & Vályi, 1996; Kurzhanski & Filippova, 1993) over a domain $[t_0, t_f]$ as an envelope enclosing an uncertain trajectory $\mathbf{x}(\cdot)$. We will use the definition given in Bethencourt and Jaulin (2014) and Le Bars et al. (2012) where a tube $[\mathbf{x}](\cdot) : \mathbb{R} \to \mathbb{IR}^n$ is an interval of two trajectories $[\mathbf{x}^-(\cdot), \mathbf{x}^+(\cdot)]$ such that $\forall t,\ \mathbf{x}^-(t) \leqslant \mathbf{x}^+(t)$. A trajectory $\mathbf{x}(\cdot)$ belongs to the tube $[\mathbf{x}](\cdot)$ if $\forall t, \mathbf{x}(t) \in [\mathbf{x}](t)$. Fig. 2 illustrates a scalar tube enclosing a trajectory $x^*(\cdot)$. For the sake of simplicity, the following tubes mentioned in Sections 2.2–3.1 will be of dimension 1, without loss of generality.

It is possible to implement a tube in several ways. A computer representation based on a set of boxes that sample the tube over time has been mentioned in Bethencourt and Jaulin (2014), Le Bars