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Brief paper

Dynamic controllers for column synchronization of rotation matrices: A QR-factorization approach☆

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ABSTRACT

In the multi-agent systems setting, this paper addresses continuous-time distributed synchronization of columns of rotation matrices. More precisely, k specific columns shall be synchronized and only the corresponding k columns of the relative rotations between the agents are assumed to be available for the control design. When one specific column is considered, the problem is equivalent to synchronization on the $(d-1)$-dimensional unit sphere and when all the columns are considered, the problem is equivalent to synchronization on $\mathsf{SO}(d)$. We design dynamic control laws for these synchronization problems. The control laws are based on the introduction of auxiliary variables in combination with a QR-factorization approach. The benefit of this QR-factorization approach is that we can decouple the dynamics for the k columns from the remaining $d-k$ ones. Under the control scheme, the closed loop system achieves almost global convergence to synchronization for quasi-strong interaction graph topologies.

1. Introduction

This paper considers multi-agent systems continuously evolving on $\mathsf{SO}(d)$, i.e., the set of $d \times d$ rotation matrices. The agents interact locally with each other and the neighborhood structure is determined by an interaction graph that is quasi-strongly connected. For such systems, we address the following synchronization problem. How to design control laws in the body fixed coordinate frames of the agents such that k specific columns of the rotation matrices asymptotically synchronize (converge to the set where they are the same and equal to the columns of a constant rotation matrix) as time goes to infinity. The problem is, in general, a synchronization problem on a Stiefel manifold. The control laws shall be designed by using the corresponding k columns of the relative rotations between the agents (and not the other columns). Such control laws can be used to solve the synchronization problem on the unit sphere; consider for example the case where satellites in space only monitor one axis of each of its neighbors. But it can also be used in problems where various degrees of rotations are available, or the problem where complete rotations are available. To solve the problem we introduce auxiliary variables and use a QR-factorization approach. The benefit of this approach is that the dynamics of the k columns considered can be decoupled from the dynamics of the remaining $d-k$ ones.

Two important special cases of the problem considered are synchronization of whole rotation matrices, i.e., synchronization on $\mathsf{SO}(d)$, and synchronization of one specific column vector, i.e., synchronization on the $(d-1)$-sphere. In these cases, for obvious reasons of applicability, the dimensions $d = 2$ and $d = 3$ have been mostly considered. The distributed synchronization problem on the unit sphere has been studied from various aspects (Li, 2015; Li & Spong, 2014; Olfati-Saber, 2006; Sarlette, 2009). Recently there have been some new developments (Markdahl & Goncalves, 2015; Markdahl, Wenjun, Hu, Hong, & Goncalves, 2016). Lageman and Sun (2016), Markdahl and Goncalves (2016) Markdahl, Thunberg, and Gonçalves (2017), Markdahl et al. (2016), and Pereira and Dimarogonas (2015). In Markdahl et al. (2017) the classical geodesic control law is studied for undirected graph topologies. Each agent moves in the tangent space in a weighted average of the directions to its neighbors. Almost global synchronization is de facto shown by a characterization of all the equilibria; the equilibria that are not in the synchronization set are shown to be unstable and the equilibria in the synchronization set are shown to be stable. The analysis can be seen to parallel the one in Tron, Afsari, and Vidal (2012) (also for undirected topologies) for the case of synchronization on $\mathsf{SO}(3)$, where intrinsic control laws are designed for almost global synchronization. For the case $d = 2$ an almost global synchronization approach has been presented for directed

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topologies and the 1-sphere (Scardovi, Sarlette, & Sepulchre, 2007). That approach is a special case of the one in Sarlette and Sepulchre (2009).

The problem of synchronization on $\mathsf{SO}(3)$ has been extensively studied (Deng, Liu, Wang, & Baras, 2016; Ren, 2010; Sarlette, Bonnabel, & Sepulchre, 2010; Thunberg, Song, Montijano, Hong, & Hu, 2014; Tron, Afsari, & Vidal, 2013; Tron & Vidal, 2014). Often the control algorithms are of gradient descent types and assume undirected topologies (Sarlette, Sepulchre, & Leonard, 2009; Thunberg, Montijano, & Hu, 2011). Local convergence results are often obtained (Thunberg, Goncalves, & Hu, 2016; Thunberg et al., 2014). If a global reference frame is used, one can show almost global convergence (Thunberg et al., 2014)—this is not allowed in the design of our control laws, only relative information is to be used. As mentioned above, Tron et al. (2012) provide a control algorithm for almost global convergence. The idea is to use so-called shaping functions where a gain constant can be chosen large enough to guarantee almost global consensus. The algorithm is defined in discrete time.

By introducing auxiliary state variables based on the QR -factorization of matrices, this work provides a dynamic feedback control algorithm for synchronization of the k first columns of the rotation matrices of the agents. The dynamics of the auxiliary variables follow a standard consensus protocol. The idea of using auxiliary or estimation variables with such dynamics is not new. Early works include Sarlette and Sepulchre (2009) and Scardovi et al. (2007), where the former addresses the 1-sphere and the latter addresses manifolds whose elements have constant norms and satisfy a certain optimality condition. Such manifolds are $\mathsf{SO}(d)$ and the Grassmann manifold Grass(k, d). If the approach in Sarlette and Sepulchre (2009) is used to synchronize k columns of the rotation matrix where $k < d-1$, then the entire relative rotations are used in the control design, which is not in general allowed in the problem considered here. In our proposed QR-factorization approach, only the corresponding k columns of the relative rotations are used in the controllers. Under the control scheme, the closed loop dynamics achieves almost global convergence to the synchronization set for quasi-strong interaction topologies.

An extended version of this manuscript, containing all the proofs, is available on arXiv, see Thunberg, Markdahl, and Goncalves (2017).

2. Preliminaries

We start this section with some set-definitions. We define the special orthogonal group

$$\mathsf{SO}(d) = \{Q \in \mathbb{R}^{d\times d} : Q^TQ = I_d, \det(Q) = 1\}$$

and set of skew symmetric matrices

$$\mathsf{so}(d) = \{\Omega \in \mathbb{R}^{d\times d} : \Omega^T = -\Omega\}.$$

The d-dimensional unit sphere is

$$\mathbb{S}^d = \{y \in \mathbb{R}^{d+1} : \|y\|_2 = 1\}.$$

The set of invertible matrices in $\mathbb{R}^{d\times d}$ is

$$\mathsf{GL}(d) = \{Q \in \mathbb{R}^{d\times d} : \det(Q) \neq 0\}.$$

We will make use of directed graphs, which have node set $\mathcal{V} = \{1, 2, \ldots, n\}$ and edge sets $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$. Such a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is quasi-strongly connected if it contains a rooted spanning tree or a center, i.e., there is one node to which there is a directed path from each other node in the graph. A directed path is a sequence of nodes such that any two consecutive nodes in the path comprises an edge in the graph. For $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ we define $\mathcal{N}_i = \{j \in \mathcal{V} : (i, j) \in \mathcal{E}\}$ for all i.

We will consider a multi-agent system with n agents. There are n coordinate systems $\mathcal{F}_i$, each of which corresponding to a unique agent i in the system. There is also a world (or global) coordinate system $\mathcal{F}_W$. At each time t, each coordinate system $\mathcal{F}_i$ is related to the global coordinate system $\mathcal{F}_W$ via a rotation $Q_i(t) \in \mathsf{SO}(d)$. This means that $Q_i(t)$ transforms vectors in $\mathcal{F}_i$ to vectors in $\mathcal{F}_W$.

For all i, let $Q_i(t, k)$ be the "tall matrix" consisting of the first k columns of $Q_i(t)$. Thus, $Q_i(t, d) = Q_i(t)$ and $Q_i(t, 1)$ is the first column of $Q_i(t)$. All the columns of $Q_i(t, k)$ are obviously mutually orthogonal and each one an element of the $(d-1)$-sphere. Let $Q_{ij}(t) = Q_i^T(t)Q_j(t)$ and $Q_{ij}(t, k) = Q_i^T(t, d)Q_j(t, k)$ for all i, j. These matrices comprise the relative transformations between the coordinate frames $\mathcal{F}_j$ and $\mathcal{F}_i$ and the k first columns thereof, respectively.

The matrix $R_i(t, d)$, or shorthand $R_i(t)$, is an element of $\mathbb{R}^{d\times d}$ for all i, t. The matrix $R_i(t, k) \in \mathbb{R}^{k\times k}$ is the upper left $k \times k$ block matrix of the matrix $R_i(t)$. These $R_i(t, k)$'s are communicated between the agents. For $R_i(t, k)$ invertible we define $R_{ij}(t, k)$ as $R_i(t, k)R_j^{-1}(t, k)$. Observe the difference of where the matrix inverse appears between $Q_{ij}(t, k)$ and $R_{ij}(t, k)$, i.e., $Q_{ij}(t, k) = Q_i^{-1}(t, d)Q_j(t, k)$, whereas $R_{ij}(t, k) = R_i(t, k)R_j^{-1}(t, k)$. Let $Q(t, k) = [Q_1^T(t, k), Q_2^T(t, k), \ldots, Q_n^T(t, k)]^T \in \mathbb{R}^{nd\times k}$ and $R(t, k) = [R_1^T(t, k), R_2^T(t, k), \ldots, R_n^T(t, k)]^T \in \mathbb{R}^{nk\times k}$.

The functions low$(\cdot)$ and up$(\cdot)$ are defined for matrices in $\mathbb{R}^{m_1\times m_2}$ for all $m_1 \geq m_2$. The function low$(\cdot)$ returns a matrix of the same dimension as the input, a matrix in $\mathbb{R}^{m_1\times m_2}$ that is, where each (i, j)-element of the matrix is equal to that of the input matrix if $i > j$ and equal to 0 if $i \leq j$. The function up$(\cdot)$ returns a matrix in $\mathbb{R}^{m_2\times m_2}$; each (i, j)-element of the matrix is equal to that of the input matrix if $i \leq j$ and equal to 0 if $i > j$.

We continue by introducing two assumptions that will be used in the problem formulation in the next section.

Assumption 1 (*Connectivity*). It holds that $\mathcal{G}$ is quasi-strongly connected.

Assumption 2 (*Dynamics*). The time evolution of the state of each agent i is given by

$$\frac{d}{dt}Q_i(t, d) = Q_i(t, d)U_i(t, d), \tag{1}$$

where $U_i(t, d) \in \mathsf{so}(d)$ and $Q_i(0, d) \in \mathsf{SO}(d)$. In particular it holds that

$$\frac{d}{dt}Q_i(t, k) = Q_i(t, d)U_i(t, k), \forall k \in \{1, 2, \ldots, d\}, \tag{2}$$

where $U_i(t, k) = U_i(t, d)[I_k, 0]^T$.

The $U_i(t, d)$'s are the controllers we are to design. An important thing to note in (1) is that $U_i(t, d)$, or rather the columns thereof, are defined in the $\mathcal{F}_i$-frames. If those would have been defined in the world frame $\mathcal{F}_W$, the agents would have needed to know their own rotations to that frame, i.e., the Q_i-matrices. Those matrices are not assumed to be available for the agents.

We let $(\mathsf{SO}(d))^n$ be the following subset of $\mathbb{R}^{nd\times d}$,

$$\{Z : Z = [Z_1^T, Z_2^T, \ldots, Z_n^T]^T, Z_i \in \mathsf{SO}(d)\ \forall i\}.$$

We let $(\mathsf{GL}(d))^n$ be the following subset of $\mathbb{R}^{nd\times d}$,

$$\{Z : Z = [Z_1^T, Z_2^T, \ldots, Z_n^T]^T, Z_i \in \mathsf{GL}(d)\ \forall i\}.$$

3. Problem formulation

The goal is to design $U_i(t, d)$ (and in particular $U_i(t, k)$) as a dynamic feedback control law such that the $Q_i(t, k)$-matrices asymptotically aggregate or converge to the synchronization set.