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Brief paper

Design of interval observers and controls for PDEs using finite-element approximations☆

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ABSTRACT

Synthesis of interval state estimators is investigated for the systems described by a class of parabolic Partial Differential Equations (PDEs). First, a finite-element approximation of a PDE is constructed and the design of an interval observer for the derived ordinary differential equation is given. Second, the interval inclusion of the state function of the PDE is calculated using the error estimates of the finite-element approximation. Finally, the obtained interval estimates are used to design a dynamic output stabilizing control. The results are illustrated by numerical experiments with an academic example and the Black–Scholes model of financial market.

1. Introduction

Model complexity is a key issue for development of control and observation algorithms. Sound, heat, electrostatics, electrodynamics, fluid flow, elasticity, or quantum mechanics, as well as the models of other physical phenomena, can be formalized similarly in terms of PDEs, whose distributed nature introduces an additional level of intricacy. That is why control and estimation of PDEs is a very popular direction of research nowadays (Barje, Achhab, & Wertz, 2013; Bredies, Clason, Kunisch, & von Winckel, 2013; Demetriou, 2004; Hasan, Aamo, & Krstic, 2016; Kamran & Drakunov, 2015; Krstic, 2009; Meurer, 2013; Nguyen, 2008; Russell, 2003; Smyshlyaev & Krstic, 2005, 2010). In this class of models, where the system state is a function of the space at each instant of time, the problem of its explicit measurement is natural, since only pointwise and discrete space measurements are realizable by a sensor (Jrgensen, Goldschmidt, & Clement, 1984; Vande Wouwer, Point, Porteman, & Remy, 2000). Frequently, in order to design a state estimator, the finite-dimensional approximation approach is used (Alvarez & Stephanopoulos, 1982; Dochain, 2000; Hagen & Mezic, 2003; Vande Wouwer & Zeitz, 2002), then the observation problem is addressed with the well-known tools available for finite-dimensional systems, while the convergence assessment has to be performed with respect to the solutions of the original distributed system.

After complexity, another difficulty for synthesis of an estimator or controller consists in the model uncertainty (unknown parameters or/and external disturbances). Presence of uncertainty implies that the design of a conventional estimator, converging to the ideal value of the state, is difficult to achieve. In this case a set-membership or interval estimation becomes more attainable: an observer can be constructed such that using the input–output information it evaluates the set of admissible values (interval) for the state at each instant of time. The interval width is proportional to the size of the model uncertainty (it has to be minimized by tuning the observer parameters). There are several approaches to design the interval/set-membership estimators (Jaulin, 2002; Kieffer & Walter, 2004; Olivier & Gouzé, 2004). This work is devoted to the interval observers (Efimov, Fridman, Raïssi, Zolghadri, & Seydou, 2012; Moisan, Bernard, & Gouzé, 2009; Olivier & Gouzé, 2004; Raïssi, Efimov, & Zolghadri, 2012; Raïssi, Videau, & Zolghadri, 2010), which form a subclass of set-membership estimators and whose design is based on the monotone systems theory (Farina & Rinaldi, 2000; Kaczorek, 2002; Smith, 1995). The idea of the interval observer design has been proposed rather recently in Gouzé, Rapaport, and Hadj-Sadok (2000), but it has already received numerous extensions for various classes of dynamical models. In the present paper an extension of this approach for the estimation of systems described by PDEs is discussed.

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An interval observer for systems described by PDEs using the finite-dimensional approximation approach has been proposed in Kharkovskaya, Efimov, Polyakov, and Richard (2016), in the present work the proofs of those results are given, with the additional design of an output stabilizing control and an application to a model of financial market. Using the discretization error estimates from Wheeler (1973), the enveloping interval for solutions of the PDE is evaluated. An interesting feature of the proposed approach is that being applied to a nonlinear PDE, assuming that all nonlinearities are bounded and treated as perturbations, then the proposed interval observer is linear and can be easily implemented providing bounds on solutions of the originally nonlinear PDE (under the hypothesis that these solutions exist). The proposed control strategy disposes a similar advantage, since it is designed for a finite-dimensional model, but guaranteeing boundedness of trajectories for an uncertain distributed dynamics.

The outline of this paper is as follows. After preliminaries in Section 2, and an introduction of the distributed system properties in Section 3, the interval observer design is given in Section 4. The design of an output control algorithm based on interval estimates is considered in Section 5. The results of numerical experiments are presented in Section 6.

2. Preliminaries

The real numbers are denoted by $\mathbb{R}$, $\mathbb{R}_+ = \{\tau \in \mathbb{R} : \tau \geq 0\}$. Euclidean norm for a vector $x \in \mathbb{R}^n$ will be denoted as $|x|$. The symbols I_n, $E_{n\times m}$ and E_p denote the identity matrix with dimension $n \times n$, the matrix with all elements equal 1 with dimensions $n \times m$ and $p \times 1$, respectively.

For two vectors $x_1, x_2 \in \mathbb{R}^n$ or matrices $A_1, A_2 \in \mathbb{R}^{n\times n}$, the relations $x_1 \leq x_2$ and $A_1 \leq A_2$ are understood elementwise. The relation $P \prec 0$ ($P \succ 0$) means that the matrix $P = P^{\mathrm{T}} \in \mathbb{R}^{n\times n}$ is negative (positive) definite. Given a matrix $A \in \mathbb{R}^{m\times n}$, define $A^+ = \max\{0, A\}$, $A^- = A^+ - A$ (similarly for vectors) and $|A| = A^+ + A^-$.

Lemma 1 (*Efimov et al., 2012*)**.** *Let $x \in \mathbb{R}^n$ be a vector variable, $\underline{x} \leq x \leq \overline{x}$ for some $\underline{x}, \overline{x} \in \mathbb{R}^n$. If $A \in \mathbb{R}^{m\times n}$ is a constant matrix, then*

$$A^+\underline{x} - A^-\overline{x} \leq Ax \leq A^+\overline{x} - A^-\underline{x}. \tag{1}$$

2.1. Nonnegative continuous-time linear systems

A matrix $A \in \mathbb{R}^{n\times n}$ is called Hurwitz if all its eigenvalues have negative real parts, and it is called Metzler if all its elements outside the main diagonal are nonnegative. Any solution of the linear system

$$\begin{aligned} \dot{x} &= Ax + B\omega(t), \ \omega : \mathbb{R}_+ \to \mathbb{R}_+^q, \\ y &= Cx + D\omega(t), \end{aligned} \tag{2}$$

with $x \in \mathbb{R}^n$, $y \in \mathbb{R}^p$ and a Metzler matrix $A \in \mathbb{R}^{n\times n}$, is elementwise nonnegative for all $t \geq 0$ provided that $x(0) \geq 0$ and $B \in \mathbb{R}_+^{n\times q}$ (Farina & Rinaldi, 2000; Kaczorek, 2002; Smith, 1995). The output solution $y(t)$ is nonnegative if $C \in \mathbb{R}_+^{p\times n}$ and $D \in \mathbb{R}_+^{p\times q}$. Such a dynamical system is called cooperative (monotone) or nonnegative if only initial conditions in $\mathbb{R}_+^n$ are considered (Farina & Rinaldi, 2000; Kaczorek, 2002; Smith, 1995).

For a Metzler matrix $A \in \mathbb{R}^{n\times n}$ its stability can be checked verifying a Linear Programming (LP) problem $A^{\mathrm{T}}\lambda < 0$ for some $\lambda \in \mathbb{R}_+^n \setminus \{0\}$, or the Lyapunov matrix equation $A^{\mathrm{T}}P + PA \prec 0$ for a diagonal matrix $P \in \mathbb{R}^{n\times n}$, $P > 0$ (in the general case the matrix P should not be diagonal). The L_1 and L_∞ gains for nonnegative systems (2) have been studied in Briat (2011) and Ebihara, Peaucelle, and Arzelier (2011), for this kind of systems these gains are interrelated. The conventional results and definitions on the L_2/L_∞ stability for linear systems can be found in Khalil (2002).

3. Distributed systems

In this section basic facts on finite-dimensional approximations of a PDE and some auxiliary results are given.

3.1. Preliminaries

If X is a normed space with norm $\|\cdot\|_X$, $\Omega \subset \mathbb{R}^n$ is an open set for some $n \geq 1$ and $\phi : \Omega \to X$, define

$$\|\phi\|^2_{L^2(\Omega,X)} = \int_\Omega \|\phi(s)\|_X^2\, ds, \ \ \|\phi\|_{L^\infty(\Omega,X)} = ess \sup_{s\in\Omega} \|\phi(s)\|_X.$$

By $L^\infty(\Omega, X)$ and $L^2(\Omega, X)$ denote the set of functions $\Omega \to X$ with the properties $\|\cdot\|_{L^\infty(\Omega,X)} < +\infty$ and $\|\cdot\|_{L^2(\Omega,X)} < +\infty$, respectively. Denote $I = [0, 1]$, let $C^k(I, \mathbb{R})$ be the set of functions having continuous derivatives through the order $k \geq 0$ on I. For any $q > 0$ and an open interval $I' \subset I$ define $W^{q,\infty}(I', \mathbb{R})$ as a subset of functions $y \in C^{q-1}(I', \mathbb{R})$ with an absolutely continuous $y^{(q-1)}$ and with $y^{(q)}$ essentially bounded on I', $\|y\|_{W^{q,\infty}} = \sum_{i=0}^{q} \|y^{(i)}\|_{L^\infty(I',\mathbb{R})}$. Denote by $H^q(I, \mathbb{R})$ with $q \geq 0$ the Sobolev space of functions with derivatives through order q in $L^2(I, \mathbb{R})$, and for $q < 0$ the corresponding dual spaces, while by $H_0^q(I, \mathbb{R})$ a closure of C^∞ functions having compact support in I with respect to the norm in $H^q(I, \mathbb{R})$.

For two functions $z_1, z_2 : I \to \mathbb{R}$ their relation $z_1 \leq z_2$ has to be understood as $z_1(x) \leq z_2(x)$ for all $x \in I$, the inner product is defined in a standard way:

$$\langle z_1, z_2\rangle = \int_0^1 z_1(x)z_2(x)dx \quad z_1, z_2 \in L^2(I, \mathbb{R}).$$

3.2. Approximation

Following Wheeler (1973), consider the following PDE with homogeneous Dirichlet boundary conditions:

$$\begin{aligned} \rho(x)\frac{\partial z(x,t)}{\partial t} &= L[x, z(x,t)] + r(x,t) \quad \forall (x,t) \in I \times (0,T), \\ z(x,0) &= z_0(x) \quad \forall x \in I, \\ 0 &= z(0,t) = z(1,t) \quad \forall t \in (0,T), \end{aligned} \tag{3}$$

where $I = [0, 1]$ and $T > 0$,

$$L(x,z) = \frac{\partial}{\partial x}\left(a(x)\frac{\partial z}{\partial x}\right) - b(x)\frac{\partial z}{\partial x} - q(x)z,$$

$r \in L^\infty(I \times [0,T], \mathbb{R})$, $a, b, q, \rho \in L^\infty(I, \mathbb{R})$ and there exist $a_0, a_1, \rho_0, \rho_1 \in \mathbb{R}_+$ such that

$$0 < a_0 \leq a(x) \leq a_1, \ \ 0 < \rho_0 \leq \rho(x) \leq \rho_1 \quad \forall x \in I,$$

and $a', b' \in L^2(I, \mathbb{R})$, where $a' = \partial a(x)/\partial x$.

Let $\Delta = \{x_j\}_{j=0}^{N'}$ for some $N' > 0$, where $0 = x_0 < x_1 < \cdots < x_{N'} = 1$, and $I_j = (x_{j-1}, x_j)$, $h_j = x_j - x_{j-1}$, $h = \max_{1\leq j\leq N'} h_j$. Let $P_s(I')$ be the set of polynomials of the degree less than $s+1$, $s > 0$ on an interval $I' \subseteq I$, then adopt the notation:

$$\begin{aligned} M^{s,\Delta} = \{&v \in C^0(I,\mathbb{R}) : v(x) = v_j(x)\ \forall x \in I_j, \\ &v_j \in P_s(I_j)\ \forall 1 \leq j \leq N'\} \end{aligned}$$

and $M = M_0^{s,\Delta} = \{v \in M^{s,\Delta} : v(0) = v(1) = 0\}$.

Introduce a bilinear form:

$$\mathcal{L}(y,v) = -\left\langle ay', v'\right\rangle - \left\langle by', v\right\rangle - \langle qy, v\rangle \quad y, v \in H^1(I, \mathbb{R}),$$

and define

$$\lambda \geq \frac{1}{2a_0}(\text{ess}\sup_{x\in I} b^2(x) - \text{ess}\inf_{x\in I} q(x)).$$