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Brief paper

Higher order moment stability region for Markov jump systems based on cumulant generating function[☆]

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ABSTRACT

This paper is concerned with the solution of higher order moment stability region for Markov jump systems with respect to cases of both known and unknown transition probabilities. By exploring the cumulant generating function, the original stochastic system with Markov jumping modes is transformed to a deterministic system. Then based on the estimation of matrix eigenvalues, the explicit solution for higher order moment stability region is expressed in terms of transition probabilities, operation modes, state transition matrix and its dimension, and the moment order. Several examples are presented to illustrate the effectiveness of the proposed method and its associated algorithm.

1. Introduction

As two classes of stochastic hybrid systems, Markov jump linear systems (MJLSs) and switched systems well depict the physical systems with variable structures, which may result from the abrupt phenomena such as component failure and parameter shift. Due to their wide applications and largely available mathematical techniques developed over last decades, many problems have been extensively studied for switched systems (Boukas, 2005; Zhang, Zhuang, & Braatz, 2016) and MJLSs (Zhang & Boukas, 2009b; Zhao, 2008). These problems include stability analysis and stabilization (Shi, Xia, Liu, & Rees, 2006; Zhang, Leng, & Colaneri, 2016), finite-time stability and controller design (Amato, Ambrosin, Ariola, & Cosentino, 2009; Luan, Min, Ding, & Liu, 2015), filter design (Wang, Shi, Wang, & Duan, 2013; Zhang & Boukas, 2009a) and so on (Luan, Zhao, & Liu, 2013; Zhang, Ning, & Shi, 2015). The techniques used in the previous literatures mainly relied on the Lyapunov–Krasovskii function and the so-called multiple model (MM) approach, which deals directly with the state of each subsystem.

Different from the MM approach, operator theory was first used in Costa, Fragoso, and Marques (2005) as a powerful tool to handle the mean square stability by formulating the systems under the Markov framework via the augmented state associated with operation modes. Then the stochastic stability and convergence rate analysis (Costa & Figueiredo, 2014; Zhao & Yong, 2010), observability and detectability (Shen, Sun, & Wu, 2013), and H_2 and quadratic control (Oswaldo & Danilo, 2016; Oswaldo, Marcelo, & Marcos, 2015) for MJLSs based on operator theory were investigated respectively. On the other hand, the comparison principle was presented in Alwan and Liu (2015) and Li and Zhu (2014) to obtain some stability criteria by reducing the dimension of the original stochastic MJLSs to one-dimensional systems.

In contrast to the existing techniques on the analysis and synthesis for MJLS, whose available results were restricted to the first order or the second order moment performance, the general higher order moment performance is investigated in this paper. In practice, for systems with wide-band perturbation noises, such as communication systems (Yüksel, 2012), chemical reacting systems (Cao, Petzold, Rathinam, & Gillespie, 2004), and randomly perturbed electrical or mechanical systems (Kozic, Janevski, & Pavlovic, 2009; Verdejo, Kliemann, & Vargas, 2014; Xie, 2006), the stability of higher order moment has to be dealt with to guarantee the performance of such systems across all frequency band. To this end, the operation properties of cumulant generating functions are utilized in this paper to reconstruct the stochastic MJLS into a deterministic linear system. The transformation of stochastic system to a deterministic one makes it possible to utilize a large

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number of design techniques for deterministic systems. Then the condition on higher order moment stability including a spectral criterion is derived. Furthermore, the explicit solution of stability region is obtained by exploring matrix eigenvalues boundary. Finally, considering the practical case that transition probability (TP) is often unknown and behaves as a random process varying with time, the stability condition is extended to MJLSs with unknown TP. Gaussian probability density function (PDF) is utilized to characterize the random TP. The effect of mean and variance of the characterized Gaussian TP on the region of higher order moment stability is discussed.

The main contribution of this paper lies in the transformation of original MJLS to a deterministic system based on the cumulant generating function. It offers a new perspective and establishes a systematic framework for investigating MJLS. Based on the cumulant generating function, the obtained deterministic system is a higher order expression with respect to the moment order, and plays an important role for systems with wide-band perturbation noises. Meanwhile, not only can the mean stability and mean square stability be considered as two special cases of higher moment stability, but also other control synthesis problems and filtering problems can be further studied based on the newly constructed deterministic model. In addition, to reveal the essential influence of transition probabilities, operation modes, moment order, and system parameter on stability, the explicit solution of the higher moment stability region is derived by estimating the eigenvalues of the system matrix.

The remainder of this paper is organized as follows. Section 2 contains the preliminary and the problem formulations. In Section 3, the main results based on the cumulant generating function and the estimation of matrix eigenvalues are obtained. Section 4 presents the analysis and discussions of the effects of certain parameters on stability region. We conclude the paper and discuss the future perspectives in Section 5.

Notation: The notations used in this paper are fairly standard. The superscript "T" stands for matrices transpose. R^n and $R^{n\times m}$ denote n dimensional Euclidean space and the set of all $n \times m$ matrices, respectively. $E\{\cdot\}$ stands for the statistical expectation of the stochastic process or vector. $|A|$ and $\|A\|$ denote the absolute value and the norm of a real matrix A, respectively. $\rho(A)$ and $\lambda(A)$ denote the spectral radius and the eigenvalues of A, respectively. Symbol $\otimes$ is the Kronecker product and I_n is the identity matrix with dimension of $n \times n$. The formula $A \otimes A$ is expressed as $A^{\otimes 2}$, and accordingly $A \otimes A \cdots \otimes A = A^{\otimes h}$.

2. Problem formulation and preliminaries

Consider the following discrete-time MJLS of the form

$$\begin{cases} x(k+1) = A_{\theta_k} x(k) \\ x(0) = x_0, \theta(0) = \theta_0 \end{cases} \tag{1}$$

where $x(k) \in R^n$ is the vector of state variables and A_{θ_k} is the known mode-dependent matrix with appropriate dimension. The random process $\{\theta_k, k \geq 0\}$ is a Markov chain taking values in a finite set $\mathbb{M} = 1, 2, \ldots, M$ with mode TP:

$$\pi_{ij} = P_r(\theta_k = j | \theta_{k-1} = i) \tag{2}$$

where π_{ij} denotes the transition probability from mode i at time $k-1$ to mode j at time k, and satisfies $\pi_{ij} \geq 0$, $\sum_{j=1}^{M} \pi_{ij} = 1, \forall\, i, j \in \mathbb{M}$.

For elements of the sigma algebra of events $\mathbb{A} \in \mathbb{R}$, the indicator function $\mathbf{1}_{\mathbb{A}}$ is defined in the usual way for $\omega \in \Omega$

$$\mathbf{1}_{\mathbb{A}}(\omega) = \begin{cases} 1 & \text{if } \omega \in \mathbb{A} \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

Define

$$q_i(k) = x(k)\mathbf{1}_{\{\theta_k = i\}}. \tag{4}$$

We have

$$\begin{aligned} q_j(k+1) &= x(k+1)\mathbf{1}_{\{\theta_{(k+1)}=j\}} \\ &= \sum_{i=1}^{M} A_i x(k) \mathbf{1}_{\{\theta_{(k+1)}=j\}} \mathbf{1}_{\{\theta_k = i\}} \\ &= \sum_{i=1}^{M} \pi_{ij} A_i q_i(k). \end{aligned} \tag{5}$$

The objective of this work is to analyze the higher order moment stability of discrete-time MJLS (1) by transforming it into a deterministic system, which leads to the following definitions:

Definition 1 (*Fang, Loparo, & Feng, 1994*; *Feng, Loparo, Ji, & Chizeck, 1992*)**.** Jump linear system (1) in the form of a Markovian process $\{\theta_k, k \geq 0\}$ is said to be h-moment stable, if for any $x_0 \in R^n$ and θ_0, the following condition holds:

$$\lim_{k\to\infty} E\left\{\|x(k)\|^h\right\} = 0.$$

Remark 1. For $h = 1$, h-moment stability reduces to $\lim_{k\to\infty} E\{\|x(k)\|\} = 0$, i.e. mean stability. For $h = 2$, h-moment stability reduces to mean square stability or second moment stability, i.e. $\lim_{k\to\infty} E\left\{\|x(k)\|^2\right\} = 0$. For $h \geq 3$, the concept of h-moment stability can be named as higher order moment stability.

Definition 2. For a random variable $z \in R^p$ with distribution density function $p(z)$, the moment generating function (MGF) is defined by $\Phi_z(\varpi) = \int_{R^p} e^{\varpi^{\mathrm{T}} z} p(z) dz$, and the cumulant generating function (CGF) is defined by $\Psi_z(\varpi) = \log \Phi_z(\varpi)$. This standard definition can be found in most textbooks and is written here for the sake of illustrating the symbol.

If MGF $\Phi_z(\varpi)$ and CGF $\Psi_z(\varpi)$ defined in Definition 2 are analytical, they can be expanded as Taylor's series at the neighborhood of $\varpi = 0$ as:

$$\Phi_z(\varpi) = \sum_{h=0}^{\infty} m(h,n)^{\mathrm{T}} \frac{\varpi^{\otimes h}}{h!} \tag{6}$$

$$\Psi_z(\varpi) = \sum_{h=0}^{\infty} c(h,n)^{\mathrm{T}} \frac{\varpi^{\otimes h}}{h!} \tag{7}$$

where $m(h,n)$ is the hth order moment vector with dimension $n^h \times 1$, given by

$$m(h,n) = \int_{R^n} z^{\otimes h} p(z) dz \tag{8}$$

$c(h,n)$ is the hth order cumulant which can be calculated by

$$c(h,n) = m(h,n) - \sum_{l=1}^{h-1} \binom{h-1}{l} K_l c(h-l,n) \otimes m(l,n) \tag{9}$$

where K_l is a specific commutation matrix with appropriate dimension which has been discussed in more detail in Rao, Terdik, and Terdik (2007).

According to Eqs. (6)–(7), taking CGF on both sides of Eq. (5), and expanding it to Taylor's series at the neighborhood of $\varpi = 0$, the left side of (5) can be described as:

$$\Psi_{q_j(k+1)} = \sum_{h=0}^{\infty} c_{q_j(k+1)}(h,n)^{\mathrm{T}} \frac{\varpi^{\otimes h}}{h!} \tag{10}$$