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Brief paper

Speeding up finite-time consensus via minimal polynomial of a weighted graph — A numerical approach☆

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ABSTRACT

This work proposes an approach to speed up finite-time consensus algorithm using the weights of a weighted Laplacian matrix. It is motivated by the need to reach consensus among states of a multi-agent system in a distributed control/optimization setting. The approach is an iterative procedure that finds a low-order minimal polynomial that is consistent with the topology of the underlying graph. In general, the lowest-order minimal polynomial achievable for a network system is an open research problem. This work proposes a numerical approach that searches for the lowest order minimal polynomial via a rank minimization problem using a two-step approach: the first being an optimization problem involving the nuclear norm and the second a correction step. Convergence of the algorithm is shown and effectiveness of the approach is demonstrated via several examples.

1. Introduction

Achieving consensus of states is a well-known important feature for networked system, see for example Olfati-Saber and Murray (2004) and Ren and Beard (2007). Many distributed control/optimization problems over a network require a consensus algorithm as a key component. The most common consensus algorithm is the dynamical system defined by the Laplacian matrix for continuous time system and the Perron matrix for discrete-time system. Past works in the general direction of speeding up convergence of these algorithms exist. For example, the work of Xiao and Boyd (2004) proposes a semi-definite programming approach to minimize the algebraic connectivity over the family of symmetric matrices that are consistent with the topology of the network. Their approach, however, results in asymptotic convergence towards the consensus value and is most suitable for large networks. More recent works focus on finite-time convergence consensus algorithm (Hendrickx, Jungers, Olshevsky, & Vankeerberghen, 2014; Hendrickx, Shi, & Johansson, 2015; Sundaram & Hadjicostis, 2007; Wang & Xiao, 2010; Yuan, Stan, Shi, Barahona, & Goncalves, 2013; Yuan, Stan, Shi, & Goncalves, 2009) which is generally preferred for small to moderate sized networks. One important area in finite-time convergence literature is the determination of the asymptotic value of a consensus network using a finite number of state measurement. Typically, the approach adopted is based on the z-transform final-value theorem and on the finite-time convergence for individual node (Sundaram & Hadjicostis, 2007; Yuan et al., 2013, 2009) without knowledge of the full network. Other works in finite-time consensus include the design of a short sequence of stochastic matrices $A_k, \dots, A_0$ such that $z(k) = \Pi_{j=1}^{k} A_j z(0)$ reaches consensus after k steps (Hendrickx et al., 2015; Ko & Shi, 2009).

Unlike past works (Sundaram & Hadjicostis, 2007; Yuan et al., 2013, 2009) where the network is unknown, this work assumed a known network and proposes an approach to speed up finite-time convergence of consensus algorithm via the choices of the weights associated with the edges of the graph. Thus, it is similar in spirit to the work of Xiao and Boyd (2004) except that the intention is to find a low-order minimal polynomial. Ideally, the lowest-order minimal polynomial should be used. However, the lowest minimal polynomial achievable for a given graph with variable weights is an open research problem (Fallat & Hogben, 2007). They are only known for some special classes of graphs (full connected, star-shaped, strongly regular and others), van Dam and Haemers (1998) and van Dam, Koolen, and Tanaka (2014). For this reason, this paper adopts a computational approach towards finding a low-order minimal polynomial. The proposed approach achieves the lowest order minimal polynomial in many of the special classes of graphs and almost always yields minimal polynomial of order lower than those obtained from standard Perron matrices of general graphs. These are demonstrated by several numerical examples.

The choice of the weights is obtained via a rank minimization problem. In general, rank minimization is a well-known difficult

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problem (Fazel, Hindi, & Boyd, 2004; Recht, Fazel, & Parrilo, 2010). Various approaches have been proposed in the literature including the nuclear norm relaxation, bilinear projection and others. This work uses a unique two-step procedure: the first is a nuclear norm optimization problem and the second, which uses the results of the first, is a correction step based on a low rank approximation. While both steps of this two-step procedure have appeared in the literature, the use of the two in a two-step predictor–corrector procedure is novel, to the best of the authors' knowledge. Hence, the proposed rank minimization approach can be of independent interest, as well as the expression of finite-time convergence value obtained via a non z-transform mechanization.

The remainder of this paper is organized as follows. This section ends with a description of the notations used. Section 2 reviews features of the standard Laplacian and Perron matrices as well as minimal polynomial and its properties. Section 3 presents the procedure of obtaining the consensus value from the minimal polynomial and discusses, in detail, the key subalgorithm used in the overall algorithm including a convergence result. The overall algorithm is described in Section 4 and the performance of the approach is illustrated via several numerical examples in Section 5. Conclusions are given in Section 6.

The notations used in this paper are standard. Non-negative and positive integer sets are indicated by $\mathbb{Z}_0^+$ and $\mathbb{Z}^+$, respectively; whereas, $\mathbb{R}$, $\mathbb{R}^n$, $\mathbb{R}^{n\times m}$ refer, respectively, to the sets of real numbers, n-dimensional real vectors and n by m real matrices. I_n is the $n\times n$ identity matrix with 1_n being the n-column vector of all ones (subscript omitted when the dimension is clear). Given a set C, $|C|$ denotes its cardinality. The transpose of matrix M and vector v are indicated by M' and v', respectively. For a square matrix Q, $Q \succ (\succeq) 0$ means Q is positive definite (semi-definite), $spec(Q)$ refers to the set of its eigenvalues, and $vec(Q)$ is the representation of elements of Q as a vector. The cones of symmetric positive semi-definite and symmetric and positive definite matrices are $\mathcal{S}_{0+}^n = \{M \in \mathbb{R}^{n\times n} | M = M', M \succeq 0\}$ and $\mathcal{S}_{+}^n = \{M \in \mathbb{R}^{n\times n} | M = M', M \succ 0\}$, respectively. The ℓ_p-norm of $x \in \mathbb{R}^n$ is $\|x\|_p$ for $p = 1, 2, \infty$ while $\|M\|_*$, $\|M\|_2$, $\|M\|_F$ are the nuclear, operator (induced) and Frobenius norm of matrix M. Diagonal matrix is denoted as $diag\{d_1, \dots, d_n\}$ with diagonal elements d_i. Additional notations are introduced when required.

2. Preliminaries and problem formulation

This section begins with a review of standard consensus algorithm and sets up the notations needed for the sequel. The network of n nodes is described by an undirected graph $G = (\mathcal{V}, \mathcal{E})$ with vertex set $\mathcal{V} = \{1, 2, \dots, n\}$ and edge set $\mathcal{E} \subseteq \mathcal{V}\times\mathcal{V}$. The pair $(i, j) \in \mathcal{E}$ if i is a neighbor of j and vice versa since G is undirected. The set of neighbors of node i is $N_i := \{j \in \mathcal{V} : (i, j) \in \mathcal{E}, i \neq j\}$. The standard adjacency matrix $\mathcal{A}_s$ of G is the $n \times n$ matrix whose (i, j) entry is 1 if $(i, j) \in \mathcal{E}$, and 0 otherwise.

The implementation of the proposed consensus algorithm is a discrete-time system of the form $z(k + 1) = \mathcal{P}z(k)$ where $\mathcal{P}$ is the Perron matrix. However, for computational expediency, the working algorithm uses the weighted Laplacian matrix $\mathcal{L} \in \mathcal{S}_{0+}^n$. The conversion of $\mathcal{L}$ to $\mathcal{P}$ is standard and is discussed later, together with desirable properties of $\mathcal{P}$ and $\mathcal{L}$. The properties of standard (non-weighted) $\mathcal{L}$ are first reviewed.

The standard Laplacian matrix $\mathcal{L}_s$ of a given G is

$$[\mathcal{L}_s]_{i,j} = \begin{cases} -1, & \text{if } j \in N_i; \\ |N_i|, & \text{if } i = j; \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

In this form, it is easy to verify that (i) eigenvalues of $\mathcal{L}_s$ are real and non-negative, (ii) eigenvectors corresponding to different eigenvalues are orthogonal, (iii) $\mathcal{L}_s$ has at least one eigenvalue 0 with eigenvector 1_n. Properties (i) and (ii) follow from the fact that $\mathcal{L}_s$ is symmetric and positive semi-definite while property (iii) is a result of the row sum of all rows being 0. Suppose the assumption (**A1**): G is connected is made. Then, it is easy to show that the eigenvalue of 0 is simple with eigenvector 1_n. Consequently, the consensus algorithm of $\dot{x}(t) = -\mathcal{L}_s x(t)$ converges to $\frac{1}{n}1_n(1_n' x(0))$.

Unlike (1), this work uses the weighted Laplacian

$$\mathcal{L}(W, G) = \mathcal{D}(G) - \mathcal{A}(G, W) \tag{2}$$

where $\mathcal{A}(G, W)$ is the weighted adjacency matrix with $[\mathcal{A}(G, W)]_{ij} = w_{ij}$ when $(i, j) \in \mathcal{E}$, $\mathcal{D}(G) = diag\{d_1, d_2, \dots, d_n\}$ with $d_i := \sum_{j\in N_i} w_{ij}$ and $W := \{w_{ij} \in \mathbb{R} | (i, j) \in \mathcal{E}\}$. The intention of this work is to compute algorithmically the minimal polynomial of $\mathcal{L}(W, G)$ over variable W for a given G. However, since the minimal polynomial attainable for a given network G is a well-known difficult problem (Fallat & Hogben, 2007), the output of the algorithm can be seen as an upper bound on the order of the achievable minimal polynomials of $\mathcal{L}(W, G)$ over all W. Note that the value of w_{ij} is arbitrary including the possibility that $w_{ij} = 0$ and $w_{ij} < 0$ for $(i, j) \in \mathcal{E}$. This relaxation allows for a larger W search space but brings about the possibility of losing connectedness of $\mathcal{L}(W, G)$ even when G is connected. Additional conditions are therefore needed to preserve connectedness, as discussed in the sequel. Since G is fixed, its dependency in $\mathcal{L}(\cdot)$, $\mathcal{D}(\cdot)$ and $\mathcal{A}(\cdot)$ is dropped for notational convenience unless required.

The desirable properties of $\mathcal{L}(W)$ are as follows:

(L1) All eigenvalues are non-negative.
(L2) 0 is a simple eigenvalue with eigenvector 1_n.
(L3) $[\mathcal{L}(W)]_{ij} = 0$ when $(i, j) \notin \mathcal{E}$.
(L4) $\mathcal{L}(W)$ has a low-order minimal polynomial.

Properties (L1) and (L2) are needed for $x(t)$ of the continuous time system $\dot{x}(t) = -\mathcal{L}x(t)$ to reach consensus while (L3) is a hard constraint imposed by the structure of G. Property (L4) determines the finite-time convergence towards consensus and is the objective of this work. With these properties, the corresponding Perron matrix is obtained from $\mathcal{P} := e^{-\epsilon\mathcal{L}}$ or $\mathcal{P} := I_n - \epsilon\mathcal{L}(W)$, with $0 < \epsilon < \frac{1}{max_i\{d_i\}}$). Then, it is easy to verify that $\mathcal{P}$ inherits from (L1)–(L4) the following properties:

(P1) All eigenvalues of $\mathcal{P}$ lie within the interval $(-1, 1]$.
(P2) 1 is a simple eigenvalue of $\mathcal{P}$ with eigenvector 1_n.
(P3) $[\mathcal{P}]_{ij} = 0$ when $(i, j) \notin \mathcal{E}$.
(P4) $\mathcal{P}$ has a low-order minimal polynomial.

The discrete-time consensus algorithm via $\mathcal{P}$ follows

$$z(k + 1) = \mathcal{P}z(k) \tag{3}$$

for discrete variable $z \in \mathbb{Z}_0^+$. From (P1), (P2) and (A1), it is easy to show, with (σ_i, ξ_i) being the ith eigenpair of $\mathcal{P}$, that $\lim_{k\to\infty} z(k) = \lim_{k\to\infty}(\sum_{i=1}^n \xi_i\xi_i'\sigma_i^k)z(0) = \frac{1}{n}1_n$. Hence, $\lim_{k\to\infty} z(k)$ reaches consensus among all its elements. The above shows that finding a $\mathcal{P}$ that possesses properties (P1)–(P4) is equivalent to finding an $\mathcal{L}(W)$ having properties (L1)–(L4). The remaining paragraphs of this section review definitions and properties of the minimal and characteristic polynomials.

Definition 1. The minimal polynomial $m_M(t)$ of a square matrix M is the monic polynomial of the lowest order such that $m_M(M) = 0$.

Several well known properties of characteristic and minimal polynomial (see for example, Friedberg, Insel, and Spence (2003) or others) are collected in the next lemma.