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Brief paper

The effects of linear and nonlinear characteristic parameters on the output frequency responses of nonlinear systems: The associated output frequency response function ☆

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ABSTRACT

In the present study, a new concept known as the Associated Output Frequency Response Function (AOFRF) is introduced to facilitate the analysis of the effects of both linear and nonlinear characteristic parameters on the output frequency responses of nonlinear systems. Based on the AOFRF concept, the study has shown, for the first time, that the output frequency responses of a wide class of nonlinear systems that are described by the NARX (Nonlinear Auto Regressive with eXogenous input) model can be represented by a polynomial function of both the system linear and nonlinear characteristic parameters of interests to the system analysis. Moreover, an efficient algorithm is derived to determine the structure and coefficients of the AOFRF based representation for system output frequency responses. Finally, a case study is provided to demonstrate the effectiveness and advantages of the new AOFRF based representation and the implication of the result to the analysis and design of nonlinear systems in the frequency domain.

1. Introduction

Because the linear system frequency domain analysis and design have a wide range of scientific and engineering applications, the analysis and design of nonlinear systems in the frequency domain have also received considerable interests (Dobrowiecki & Schoukens, 2007). Compared with the time domain methods such as, e.g., the traditional harmonic balance and multi-scale methods, the frequency domain approaches have shown the capability to deal with a wide class of nonlinear systems, rather than the systems with specific model descriptions (Novara, Fagiano, & Milanese, 2013).

Based on the Volterra series theory of nonlinear systems, the concept of Generalized Frequency Response Functions (GFRFs) was proposed in 1959 (George, 1959), which are a series of multi-dimensional functions. The multi-dimensional nature makes the GFRFs difficult to be applied in practice. To address this challenge, some one-dimensional frequency domain representations of nonlinear systems have been proposed. These include, for example, Nonlinear Output Frequency Response Functions (NOFRFs) (Lang & Billings, 2005), Output Frequency Response Function (OFRF) (Lang, Billings, Yue, & Li, 2007), Higher Order Sinusoidal Input Describing Functions (HOSIDF) (Nuij, Bosgra, & Steinbuch, 2006), and the nonlinear Bode plots (Pavlov, Van de Wouw, & Nijmeijer, 2007). The NOFRFs represent the relationship between the input and output spectra of nonlinear systems in a way similar to the Frequency Response Function (FRF) based representation of linear systems and have found applications in the areas of structural health monitoring and fault diagnosis (Zhao et al., 2015). The OFRF reveals an analytical relationship between the output spectrum of nonlinear systems and the parameters which define system nonlinearities, and provide an effective approach to the design of the system nonlinear properties in the frequency domain (Ho, Lang, & Billings, 2014). The HOSIDF can be considered to be a special case of the OFRF of a static polynomial nonlinear system (Rijlaarsdam, Nuij, Schoukens, & Steinbuch, 2011). Considering the wide application of the FRF in linear system analysis, Pavlov, Van de Wouw and Nijmeijer (2007) proposed the concept of nonlinear Bode plots for the analysis and design of nonlinear convergent systems. However, in most cases, the nonlinear Bode plots cannot be analytically studied (Rijlaarsdam, Nuij, Schoukens, & Steinbuch, 2017) and are, consequently, difficult to be used to understand the properties of underlying systems.

It is well known that the output frequency responses of nonlinear systems are affected by both the linear and nonlinear characteristic parameters of the system. The OFRF shows an analytical

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relationship between the output spectra of nonlinear systems and the system's nonlinear characteristic parameters, but this relationship is only valid under the condition that the system linear characteristic parameters are fixed. Very recently, the issue associated with the effect of linear characteristic parameters on the nonlinear system output frequency responses have been studied (Xiao & Jing, 2016). However, the result is, so far, only a conceptual polynomial approximation for the system output spectrum, and there are still no results that can systematically relate the output frequency response of nonlinear systems to both system linear and nonlinear characteristic parameters so as to facilitate the system analysis and design.

In the present study, motivated by the need of the analysis and design of the effects of any parameters of a nonlinear system on the output frequency response, a new concept known as Associated Output Frequency Response Function (AOFRF) is introduced for the NARX model of nonlinear systems. Based on the novel AOFRF concept, it is rigorously shown that the output frequency response of nonlinear systems can be represented by a polynomial function of both the system linear and non-linear characteristic parameters. Effective algorithms are derived to determine the structure and coefficients of the AOFRF based representation of the output frequency response of nonlinear systems. Finally, a case study is used to show the application and verify the effectiveness of the algorithms in the analysis of output frequency responses of nonlinear systems to both deterministic and random inputs. The results demonstrate the significance of the new AOFRF concept and associated techniques in the revelation of the effects of both linear and nonlinear characteristic parameters on the output frequency responses of a wide class of nonlinear systems.

2. The output frequency responses of nonlinear systems

Consider the nonlinear systems described by a polynomial NARX model (Peyton-Jones & Billings, 1989)

$$y(k)=\sum_{m=1}^{M}\sum_{p=0}^{m}\sum_{k_1,k_{p+q}=1}^{K}\left[c_{p,q}\left(k_1,\ \ldots,\ k_{p+q}\right)\prod_{i=1}^{p}y(k-k_i)\times\prod_{i=p+1}^{p+q}u(k-k_i)\right]\tag{1}$$

where $y(.)$ and $u(.)$ are the outputs and inputs of the system; k represents the discrete time; $c_{p,q}\left(k_1,\ \ldots,\ k_{p+q}\right)$ with $p+q=m$ represents the model coefficients of the NARX model and $\sum_{k_1,k_{p+q}=1}^{K}=\sum_{k_1=1}^{K}\cdots\sum_{k_{p+q}=1}^{K}$; M and K are integers.

The NARX model (1) is a deterministic representation of nonlinear systems. When the physical model of nonlinear systems under study is not available, a NARX model can be obtained from a NARMAX (Nonlinear Auto Regressive Moving Average with eXogenous input) model by removing the terms representing modeling error and noise in the model. The NARMAX model can be identified from the system input/output data using the NARMAX method of nonlinear system identification (Billings, 2013).

Under the condition that system (1) is stable at zero equilibrium, the output of system (1) can be described by the discrete time Volterra series

$$y(k)=\sum_{n=1}^{N}y_n(k)=\sum_{n=1}^{N}\sum_{\tau_1=-\infty}^{+\infty}\cdots\sum_{\tau_n=-\infty}^{+\infty}h_n(\tau_1,\ldots,\tau_n)\prod_{i=1}^{n}u(k-\tau_i)\tag{2}$$

where $h_n(\tau_1,\ldots,\tau_n)$ is the nth order Volterra kernel, and N is the maximum nonlinear order of the Volterra series. Moreover, the output spectrum of the system can be represented as (Lang & Billings, 1996)

$$Y(\mathrm{j}\omega)=\sum_{n=1}^{N}Y_n(\mathrm{j}\omega)=\sum_{n=1}^{N}\frac{1}{\sqrt{n}(2\pi)^{n-1}}\times\int_{\omega_1+\cdots+\omega_n=\omega}H_n(\omega_1,\ldots,\omega_n)\prod_{i=1}^{n}U(\mathrm{j}\omega_i)\,\mathrm{d}\sigma_\omega\tag{3}$$

where $-\pi f_s\le\omega\le\pi f_s$, $f_s=1/\Delta t$ is the sampling frequency; $U(\mathrm{j}\omega)$ and $Y(\mathrm{j}\omega)$ are the spectra of the system input and output, respectively, and

$$H_n(\omega_1,\ldots,\omega_n)=\sum_{\tau_1=-\infty}^{+\infty}\cdots\sum_{\tau_n=-\infty}^{+\infty}h_n(\tau_1,\ldots,\tau_n)\times\exp\left(-\mathrm{j}(\omega_1\tau_1+\cdots+\omega_n\tau_n)\Delta t\right)\tag{4}$$

is the nth order GFRF of the system.

According to the concept of the OFRF proposed by Lang, Billings, Yue, and Li (2007), the output spectrum of a wide class of nonlinear systems can rigorously be represented by a polynomial function of the system nonlinear characteristic parameters.

Recently, Xiao and Jing (2016) have conceptually indicated that the output spectrum can also be represented by a polynomial function of the system linear characteristic parameters. These results imply that the output spectrum of nonlinear systems could be represented by a polynomial function of both the system linear and nonlinear characteristic parameters. However, there are neither results about the conditions under which this representation is valid nor algorithms that can be used to determine the detailed structure of this polynomial.

The present study is motivated by the successful application of the OFRF and associated techniques (Ho, Lang & Billings, 2014; Lang, Guo, & Takewaki, 2013; Lang, Jing, Billings, Tomlinson, & Peng, 2009) and the need to study the effects of both the linear and nonlinear characteristic parameters on the output responses of nonlinear systems. The work will be based on a new concept known as the Associated Output Frequency Response Function (AOFRF).

3. The concept of the Associated Output Frequency Response Function (AOFRF)

The concept of the AOFRF of the NARX model of nonlinear systems is introduced in Proposition 1 to facilitate the representation of the system output frequency response in terms of both the NARX model linear and nonlinear characteristic parameters.

Proposition 1. *The output spectrum of the nonlinear system (1)/(2) can be described as*

$$Y(\mathrm{j}\omega)=\sum_{r=0}^{N}\widetilde{Y}_r(\mathrm{j}\omega)\tag{5}$$

where

$$\widetilde{Y}_r(\mathrm{j}\omega)=\sum_{n=r}^{N}\int_{\omega_1+\cdots+\omega_n=\omega}H_1^r(\omega_1,\ldots,\omega_r)\times\left[\boldsymbol{L}_{(n:r)}(\omega_1,\ldots,\omega_n)\circ\boldsymbol{\Phi}_{(n:r)}(\omega_1,\ldots,\omega_n)\right]\boldsymbol{C}^{\mathrm{T}}{}_{(n:r)}\,\mathrm{d}\sigma_\omega\tag{6}$$

is referred to as the rth order AOFRF with $r=0,\ldots,n$; "∘" represents the Hadamard product,

$$\boldsymbol{\Phi}_{(n:r)}(\omega_1,\ldots,\omega_n)=\frac{\boldsymbol{\Phi}_{H(n:r)}(\omega_1,\ldots,\omega_n)\prod_{j=1}^{n}U\left(\mathrm{j}\omega_j\right)}{\sqrt{n}(2\pi)^{n-1}}\tag{7}$$