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A note on establishing convergence in adaptive systems ☆

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ABSTRACT

We present a new formulation of a convergence result for Lyapunov function candidates satisfying a differential inequality with integrable coefficients that often appears in adaptive control problems. Usually, Barbalat's Lemma is invoked, requiring boundedness of the time derivative of the Lyapunov function candidate which can sometimes be hard to establish. By connecting results from the literature, an alternative route avoiding Barbalat's Lemma is suggested.

1. Introduction

Consider the adaptive control problem of regulating the scalar state x of the system

$$\dot{x} = ax + u \tag{1}$$

to zero, where a is an unknown constant and u is the control input. Following a standard identifier-based approach to design u, we select the identifier

$$\dot{\hat{x}} = -\gamma_0(\hat{x} - x) + \hat{a}x + u + k_0(x - \hat{x})x^2 \tag{2}$$

where γ_0 and k_0 are positive design gains. The error $e = x - \hat{x}$ satisfies

$$\dot{e} = -\gamma_0 e + \tilde{a}x - k_0 e x^2 \tag{3}$$

where the parameter estimation error $\tilde{a} = a - \hat{a}$ has been defined. Consider the Lyapunov function candidate V_1, defined as

$$V_1 = \frac{1}{2}e^2 + \frac{1}{2\gamma_1}\tilde{a}^2 \tag{4}$$

for some design scalar $\gamma_1 > 0$. Differentiating (4) with respect to time and inserting the dynamics (3), we obtain

$$\dot{V}_1 = -\gamma_0 e^2 - k_0 e^2 x^2 \tag{5}$$

where we have chosen the adaptive law

$$\dot{\hat{a}} = \gamma_1 e x. \tag{6}$$

From (5) it is clear that V_1 is non-increasing, and therefore

$$e, \tilde{a} \in \mathcal{L}_\infty \quad \text{(bounded)}. \tag{7}$$

Since V_1 is non-increasing and bounded from below, V_1 has a limit as $t \to \infty$, and so (5) can be integrated from $t = 0$ to infinity to obtain

$$e, ex \in \mathcal{L}_2 \quad \text{(square-integrable)}. \tag{8}$$

Now, choosing the control law

$$u = -\hat{a}x - \gamma_2\hat{x} \tag{9}$$

for a design gain $\gamma_2 > 0$, and substituting into (2), we get

$$\dot{\hat{x}} = -\gamma_2\hat{x} + \gamma_0 e + k_0 e x^2. \tag{10}$$

Consider the Lyapunov function candidate

$$V_2 = \frac{1}{2}\hat{x}^2 + \frac{1}{2}e^2. \tag{11}$$

Differentiating (11) with respect to time and inserting the dynamics (3) and (10), and using Young's inequality, yield

$$\begin{aligned}\dot{V}_2 &= -\gamma_2\hat{x}^2 + \hat{x}\gamma_0 e + k_0\hat{x}ex^2 - \gamma_0 e^2 + e\tilde{a}x - k_0 e^2 x^2 \\ &\leq -\gamma_2\hat{x}^2 + \frac{\rho_1\gamma_0\hat{x}^2}{2} + \frac{\gamma_0 e^2}{2\rho_1} + \frac{k_0\rho_2\hat{x}^2 e^2 x^2}{2} + \frac{k_0\hat{x}^2}{\rho_2} \\ &\quad + \frac{k_0 e^2}{\rho_2} - \gamma_0 e^2 + \frac{\rho_3 e^2}{2} + \frac{\tilde{a}^2\hat{x}^2}{\rho_3} + \frac{\tilde{a}^2 e^2}{\rho_3} - k_0 e^2 x^2\end{aligned} \tag{12}$$

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for arbitrary positive constants ρ_1, ρ_2, ρ_3. Choosing $\rho_1 = \frac{\gamma_2}{3\gamma_0}$, $\rho_2 = \frac{6}{\gamma_2 k_0}$, $\rho_3 = \frac{6a_0^2}{\gamma_2}$, where a_0 upper bounds $|\hat{a}|$, and recalling that $e, ex \in \mathcal{L}_2$, we obtain

$$\dot{V}_2 \leq -cV_2 + l_1 V_2 + l_2 \tag{13}$$

where $c = \min\{\gamma_2, 2\gamma_0\}$ is a positive constant and

$$l_1 = \frac{6}{\gamma_2} e^2 x^2 \tag{14a}$$

$$l_2 = \left(\frac{3}{2}\frac{\gamma_0^2}{\gamma_2} + \frac{\gamma_2 k_0^2}{6} + \frac{3a_0^2}{\gamma_2} + \frac{\gamma_2}{6}\right) e^2 \tag{14b}$$

are integrable functions (i.e. $l_1, l_2 \in \mathcal{L}_1$).

At this point it is customary to set the stage for applying Barbalat's Lemma by invoking the following result:

Lemma 1 (*Lemma B.6 from Krstić, Kanellakopoulos, & Kokotović, 1995*). *Let $v(t), l_1(t), l_2(t)$, be real-valued functions defined for $t \geq 0$. Suppose,*[1]

$$v(t), l_1(t), l_2(t) \geq 0, \ \forall t \geq 0 \tag{15a}$$

$$l_1, l_2 \in \mathcal{L}_1 \tag{15b}$$

$$\dot{v}(t) \leq -cv(t) + l_1(t)v(t) + l_2(t) \tag{15c}$$

where c is a positive constant. Then

$$v \in \mathcal{L}_1 \cap \mathcal{L}_\infty. \tag{16}$$

To apply Barbalat's Lemma (Lemma 4 or Corollary 5 in the Appendix) for concluding $V_2 \to 0$, one must in addition to (16), establish that $\dot{V}_2 \in \mathcal{L}_\infty$, which happens to be the case in this example. Another option is to use Lemma 3.1 from Liu and Krstić (2001) (Lemma 6 in the Appendix), which requires $\ddot{V}_2$ to be bounded from above and not necessarily from below.

It turns out, however, that the conditions of Lemma 1 are sufficient to obtain convergence without requiring any form of boundedness on $\dot{V}_2$, a fact that follows trivially from combining Lemma 1 and the following Lemma.

Lemma 2 (*Lemma 2.17 from Tao, 2003*). *Consider a signal g satisfying*

$$\dot{g}(t) = -ag(t) + bh(t) \tag{17}$$

for a signal $h \in \mathcal{L}_1$ and some constants $a > 0, b > 0$. Then

$$g \in \mathcal{L}_\infty \tag{18}$$

and

$$\lim_{t\to\infty} g(t) = 0. \tag{19}$$

2. Extension of Lemma 1

We will here state the main point of this note, which is an extension of Lemma 1.

Lemma 3. *Let $v(t), l_1(t), l_2(t)$, be real-valued functions defined for $t \geq 0$. Suppose*

$$v(t), l_1(t), l_2(t) \geq 0, \ \forall t \geq 0 \tag{20a}$$

$$l_1, l_2 \in \mathcal{L}_1 \tag{20b}$$

$$\dot{v}(t) \leq -cv(t) + l_1(t)v(t) + l_2(t) \tag{20c}$$

where c is a positive constant. Then

$$v \in \mathcal{L}_1 \cap \mathcal{L}_\infty \tag{21}$$

and

$$\lim_{t\to\infty} v(t) = 0. \tag{22}$$

Proof. Property (21) follows from Lemma 1. Writing (20c) as

$$\dot{v}(t) \leq -cv(t) + f(t) \tag{23}$$

where

$$f(t) = l_1(t)v(t) + l_2(t) \tag{24}$$

satisfies $f \in \mathcal{L}_1$ and $f(t) \geq 0, \ \forall t \geq 0$ since $l_1, l_2 \in \mathcal{L}_1, l_1(t), l_2(t) \geq 0, \ \forall t \geq 0$ and $v \in \mathcal{L}_\infty$. Lemma 2 can be invoked for (23) with equality. The result (22) then follows from the comparison lemma.

An alternative, direct proof of (22) goes as follows. For (22) to hold, we must show that for every $\epsilon_1 > 0$, there exists $T_1 > 0$ such that

$$v(t) < \epsilon_1 \tag{25}$$

for all $t > T_1$. We will prove that such a T_1 exists by constructing it. Since $f \in \mathcal{L}_1$, there exists $T_0 > 0$ such that

$$\int_{T_0}^{\infty} f(s)ds < \epsilon_0 \tag{26}$$

for any $\epsilon_0 > 0$. Solving

$$\dot{w}(t) = -cw(t) + f(t), \tag{27}$$

and applying the comparison principle, gives the following bound for $v(t)$

$$v(t) \leq v(0)e^{-ct} + \int_0^t e^{-c(t-\tau)} f(\tau)d\tau. \tag{28}$$

Splitting the integral at $\tau = T_0$ gives

$$\begin{aligned} v(t) &\leq v(0)e^{-ct} + e^{-c(t-T_0)} \int_0^{T_0} e^{-c(T_0-\tau)} f(\tau)d\tau \\ &\quad + \int_{T_0}^t e^{-c(t-\tau)} f(\tau)d\tau \\ &\leq Me^{-ct} + \int_{T_0}^t f(\tau)d\tau \end{aligned} \tag{29}$$

for $t > T_0$, where

$$\begin{aligned} M &= v(0) + e^{cT_0} \int_0^{T_0} f(\tau)d\tau \\ &\leq v(0) + e^{cT_0} \|f\|_1 \end{aligned} \tag{30}$$

is a finite, positive constant. Using (26) with

$$\epsilon_0 = \frac{1}{2}\epsilon_1, \tag{31}$$

we have

$$\begin{aligned} v(t) &\leq Me^{-ct} + \int_{T_0}^t f(\tau)d\tau < Me^{-ct} + \epsilon_0 \\ &< Me^{-ct} + \frac{1}{2}\epsilon_1. \end{aligned} \tag{32}$$

Now, choosing T_1 as

$$T_1 = \max\left\{T_0, \frac{1}{c}\log\left(\frac{2M}{\epsilon_1}\right)\right\} \tag{33}$$

we obtain

$$v(t) < \frac{1}{2}\epsilon_1 + \frac{1}{2}\epsilon_1 = \epsilon_1 \tag{34}$$

for all $t > T_1$, which proves (22).

[1] In Krstić et al. (1995) $v(0) \geq 0$ is assumed rather than $v(t) \geq 0$.

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