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Stochastic stability in Max-Product and Max-Plus Systems with Markovian Jumps[☆]

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ABSTRACT

We study Max-Product and Max-Plus Systems with Markovian Jumps and focus on stochastic stability problems. At first, a Lyapunov function is derived for the asymptotically stable deterministic Max-Product Systems. This Lyapunov function is then adjusted to derive sufficient conditions for the stochastic stability of Max-Product systems with Markovian Jumps. Many step Lyapunov functions are then used to derive necessary and sufficient conditions for stochastic stability. The results for the Max-Product systems are then applied to Max-Plus systems with Markovian Jumps, using an isomorphism and almost sure bounds for the asymptotic behavior of the state are obtained. A numerical example illustrating the application of the stability results on a production system is also given.

1. Introduction

Max-Plus systems are dynamical systems which satisfy the superposition principle in the Max-Plus algebra. The use of Max-Plus systems was proposed in various applications involving timing, such as communication and traffic management, queueing systems, production planning, multi-generation energy systems, etc. (e.g. Baccelli, Cohen, Olsder, & Quadrat, 1992; Baccelli & Hong, 2000b; Cuninghame-Green, 1979; Goverde, 2007; Heidergott, Olsder, & Van Der Woude, 2014). Recently, the use of the closely related class of Max-Product systems (systems which satisfy the superposition principle in the Max-Product algebra) was proposed as a tool for the modeling of cognitive processes, such as detecting audio and visual salient events in multimodal video streams (Maragos & Koutras, 2015). Max-Plus and Max-Product algebras have also computational uses involving Optimal Control problems (McEneaney, 2006) and estimation problems in probabilistic models such as the max-sum algorithm in Probabilistic Graphical models and the Viterbi algorithm in Hidden Markov Models (e.g. Bishop, 2006).

In this work, we study stochastic Max-Plus and Max-Product systems, where the system matrices depend on a finite state Markov chain. For the Max-Plus systems we focus on the asymptotic growth rate, whereas for the Max-Product systems on stochastic stability. A motivation to study Max-Plus systems with Markovian jumps is to model production systems, where the processing or holding times are random variables (not necessarily independent) or there are random failures and repairs, modeled as a Markov chain. The results on max-product stochastic systems will be used as an intermediate step. An independent motivation to study Max-Product systems is the modeling of cognitive processes interrupted by random events. Similar problems with Markovian delays for linear systems were studied in Beidas and Papavassilopoulos (1993), for random failures in Papavassilopoulos (1994) and for nonlinear time varying systems in Beidas and Papavassilopoulos (1995), in the context of distributed parallel optimization and routing applications. In the current work, we try to exploit the special (Max-Product or Max-Plus) structure of the system.

At first, deterministic Max-Product systems are considered and their asymptotic stability is characterized using Lyapunov functions. The Lyapunov function derived can be also used to study systems which are not linear in the Max-Product algebra. We then study Max-Product systems with Markovian Jumps and derive sufficient conditions for their stochastic stability. Further, necessary and sufficient conditions for the stochastic stability of Max-Product systems with Markovian Jumps are derived using many step Lyapunov functions. The results for the stochastic stability of Max-Product systems are then used to derive bounds for the evolution of the state of Max-Plus systems with Markovian Jumps.

The results of this work relate to the literature for the approximation of the Lyapunov exponent of Max-Plus stochastic systems.

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The existence of the Lyapunov exponent was proved in Cohen (1988). Limit theorems for the scaled asymptotic evolution of stochastic Max-Plus systems were proved in Merlet (2007, 2008). Most of the works on the approximation of the Lyapunov exponent focus on the independent random matrix case. In Baccelli and Hong (2000a) and Gaubert and Hong (2000) series expansions are used in order to approximate the Lyapunov exponent and Goverde, Heidergott, and Merlet (2008, 2011) use approximate stochastic simulation techniques to estimate the Lyapunov exponent. In Blondel, Gaubert, and Tsitsiklis (2000) it is shown that the approximation of the Lyapunov exponent is an NP-hard problem. Bounds for the tail distributions of Max-Plus stochastic systems are proposed in Chang (1996). In Liu, Nain, and Towsley (1995), a model of Max-Plus system with Markovian input is considered and bounds for the tail distributions are derived. A model where the Markov chain (branching process) evolves according to a Max-Plus stochastic system is analyzed in Altman and Fiems (2012). Bounds on the length of the transient phase of Max-Plus systems are proved in Nowak and Charron-Bost (2014).

Another related class of systems is Switching Max-Plus systems with deterministic or stochastic switching introduced in van den Boom and De Schutter (2006) and studied further in van den Boom and De Schutter (2012). The basic difference with the current work is that the current work focuses on stochastic stability properties whereas (van den Boom & De Schutter, 2006, 2012) study stability under arbitrary switching. Several approximation methods in stochastic Max-plus systems control and identification were studied in Farahani (2012).

The techniques used in this work closely parallel the techniques used for the stability analysis of Markovian Jump Linear Systems (MJLS). The study of the stochastic stability of MJLS dates back at least to the 1960s (Bhaurucha, 1961) and today is a well-established field (e.g. Beidas & Papavassilopoulos, 1993; Costa, Fragoso, & Marques, 2006; Fang & Loparo, 2002; Kordonis & Papavassilopoulos, 2014; Papavassilopoulos, 1994).

1.1. Background

The Max-Plus and Max-Product algebras are used. In the Max-Plus algebra the usual summation is substituted by maximum and the usual multiplication is substituted by summation. In the Max-Product algebra the usual summation is substituted by maximum but the multiplication remains unchanged.

The Max-Plus algebra is defined on the set of extended reals $\bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ with the binary operations "$\oplus$" and "$\otimes$". The operation "$\oplus$" stands for the maximum i.e., for $x, y \in \bar{\mathbb{R}}$, it holds $x \oplus y = \max\{x, y\}$. The operation "$\otimes$" corresponds to the usual addition i.e., for $x, y \in \bar{\mathbb{R}}$ it holds $x \otimes y = x + y$, where the convention $-\infty \otimes \infty = -\infty$ is used. For a set $(x_i)_{i\in I}$ of extended reals "$\bigoplus$" stands for the supremum i.e. $\bigoplus_{i\in I} x_i = \sup_{i\in I}\{x_i\}$. For a pair of matrices $\boldsymbol{A} = [A_{ij}]$ and $\boldsymbol{B} = [B_{ij}]$, the operation "$\oplus$" is their element-wise maximum, i.e.:

$$(\boldsymbol{A} \oplus \boldsymbol{B})_{ij} = A_{ij} \oplus B_{ij},$$

and similarly is the element-wise supremum for an arbitrary set of matrices.

For a pair of matrices $\boldsymbol{A} = [A_{ij}] \in \bar{\mathbb{R}}^{n\times m}$ and $\boldsymbol{B} = [B_{ij}] \in \bar{\mathbb{R}}^{m\times l}$ their Max-Plus product $\boldsymbol{A}\otimes\boldsymbol{B}$ is an $n \times l$ matrix and its i, jth element is given by:

$$(\boldsymbol{A} \otimes \boldsymbol{B})_{ij} = \bigoplus_{p=1}^{m} \left(A_{ip} + B_{pj}\right), \tag{1}$$

where "$\bigoplus$" denotes the maximum of the m elements.

The Max-Product algebra is defined on $\bar{\mathbb{R}}_+ = [0, \infty]$, with the binary operations "$\oplus$" and "$\odot$". The "$\odot$" operation is the usual scalar multiplication with the convention $0 \odot \infty = 0$. The "$\oplus$" operation is defined exactly as in the Max-Plus algebra. The matrix multiplication in the Max-Product algebra is defined by:

$$[\boldsymbol{A} \odot \boldsymbol{B}]_{ij} = \bigoplus_{p=1}^{m} \left(A_{ip}B_{pj}\right).$$

The power of a square matrix is defined by $\boldsymbol{A}^k = \boldsymbol{A}^{k-1} \odot \boldsymbol{A}$ and $\boldsymbol{A}^0 = \boldsymbol{I}$. For a given square matrix $\boldsymbol{A}$ a new matrix $\boldsymbol{A}^+$ is defined as $\boldsymbol{A}^+ = \bigoplus_{k=0}^{\infty} \boldsymbol{A}^k$. The subset $\mathbb{R}_+ = [0, \infty)$ of $\bar{\mathbb{R}}_+$ will be also used.

Max-Product multiplication distributes over "$\bigoplus$", i.e.:

$$\bigoplus_{i\in I} \boldsymbol{A} \odot \boldsymbol{B}_i = \boldsymbol{A} \odot \left(\bigoplus_{i\in i} \boldsymbol{B}_i\right). \tag{2}$$

The same property holds also for the Max-Plus multiplication.

In both algebras, the "$\oplus$" operation has lower priority than "$+$" or "$\otimes$" in the Max-Plus algebra and "$\cdot$" or "$\odot$" in the Max-Product algebra respectively. Let us note that there is an isomorphism $\exp(\cdot)$ between the Max-Plus algebra $(\bar{\mathbb{R}}, \oplus, \otimes)$ and the Max-Product algebra $(\bar{\mathbb{R}}_+, \oplus, \odot)$. The notation used in this paper is summarized in Table 1.

Table 1
The algebraic operations used.

Operation	Meaning
$\oplus$	The maximum. Applies for scalars, vectors and matrices
$\otimes$	Max-plus multiplication. Defined in (1)
$\odot$	Max-plus multiplication. Defined in (2)

A unifying algebraic framework to study Max-Plus and Max-Product systems (and also other systems) is the theory of Weighted Lattices (Maragos, 2013, 2017).

1.2. Notation

For a pair of vectors $\boldsymbol{x} = (x_1, \dots, x_n)^T$ and $\boldsymbol{y} = (y_1, \dots, y_n)^T$, the inequality notation $\boldsymbol{x} \le \boldsymbol{y}$ is used meaning that $x_i \le y_i$, for all i. Similarly, the inequality notation $\boldsymbol{x} < \boldsymbol{y}$ stands for $x_i < y_i$, for all i. The infinity norm will be used i.e. $\|\boldsymbol{x}\| = \max_i |x_i|$. We denote by $\mathbf{1}$ the column vector of dimension n consisting of ones. The underlying probability space is denoted by $(\Omega, \mathcal{F}, P)$.

A function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ will be called class $\mathcal{K}$ function if α is increasing and $\alpha(0) = 0$. A function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ will be called class $\mathcal{KL}$ function if, for each fixed t, the function $\beta(\cdot, t)$ is a class $\mathcal{K}$ function and for any fixed s, the function $\beta(s, \cdot)$ is decreasing and $\beta(s, t) \to 0$ as $t \to \infty$.

1.3. Problem formulation

The first class of systems considered is Max-Product systems with Markovian jumps. The uncertainty of the system is described by a Markov chain y_k having a finite state space $\{1, \dots, M\}$ and transition probabilities c_{ij}. That is, the evolution of y_k is described by $c_{ij} = P(y_{k+1} = j | y_k = i)$. A Max-Product system with Markovian jumps is described by:

$$\boldsymbol{x}_{k+1} = \boldsymbol{A}(y_k) \odot \boldsymbol{x}_k, \qquad \boldsymbol{x}_0 \in \mathbb{R}_+^n. \tag{3}$$

That is, at each time step the system matrix A takes one of the M different values $A(1), \dots, A(M)$ according to the value of the Markov chain.

At first, the class of deterministic Max-Product systems will be considered. In these systems the matrix $\boldsymbol{A}(\cdot)$ does not depend on the Markov chain and takes a single value $\boldsymbol{A}$.