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Brief paper

Stochastic Lyapunov functions without differentiability at supposed equilibria[☆]

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ABSTRACT

Stochastic calculus indicates that allowing a Lyapunov candidate function to be non-differentiable at the origin helps its expected value to decrease. To utilize this observation, this paper investigates stability of stochastic systems, supposing that the origin of the state space is an equilibrium before influenced by Gaussian white noises. For identifying such stability properties, notions of instantaneous points and almost sure equilibria, which are mutually exclusive, are introduced. This paper clarifies the relationship between the stability properties, and develops their Lyapunov-type characterizations without differentiability at the origin. Discussions are given in relation to the notion of noise-to-state stability, and this paper proposes a method to address both transient and global-in-time properties simultaneously with a single positive definite function even when stochastic noises prevent the origin from being an equilibrium.

1. Introduction

Various stability notions have been introduced for stochastic systems such as stability in probability, moment stability, almost sure stability (Bardi & Cesaroni, 2005; Khasminskii, 2012; Mao, 2007) and asymptotic versions of them. These notions extend the Lyapunov stability theory of deterministic systems, and they coincide with corresponding deterministic notions in the absence of stochastic noises. Such stochastic stability notions are properties of equilibria.

Stability in probability can be ensured by the existence of a stochastic Lyapunov function (SLF) (Khasminskii, 2012; Mao, 2007). Moment stability and almost sure stability can also be ensured under appropriate additional conditions (Bardi & Cesaroni, 2005; Khasminskii, 2012; Mao, 2007). The stochastic stability analysis replaces a differential operator in time by an infinitesimal operator since stochastic processes such as the Wiener process have unbounded variations preventing their time derivative from existing (Itô, 2004; Øksendal, 2013). The infinitesimal operator tells that the dynamics of the expectation of an SLF involves not only the first partial derivatives of the SLF, but also the second derivatives. Therefore, twice differentiability has been assumed commonly for SLFs in the literature. However, the twice differentiability is not necessary for stability in probability (Kushner, 1967a), as shown in Remark 5.5 of Khasminskii (2012). This naturally leads one to the idea of allowing SLFs to be non-differentiable.

The first objective of this paper is to investigate issues arising from SLFs non-differentiable at the origin. Interestingly, systems often admit SLFs non-differentiable at the origin when stochastic noises disqualify the origin as an equilibrium. The non-differentiability can let the sign of the second derivative of SLFs negative, although the noises remaining at the origin make the behavior of the process spiky at the origin. Thus, employing the non-differentiability brings unique issues not present in the deterministic case. To identify stability properties for the origin, this paper classifies the origin into two groups: instantaneous points and almost sure equilibria. Based on the classification, this paper introduces new notions of transient-type stability properties. The second main objective of this paper is to develop sufficient conditions for the properties as well as stability in probability, moment stability and their asymptotic versions via SLFs without requiring differentiability at the origin.

The final objective of this paper is to strengthen the benefit of using SLFs non-differentiable at the origin by securing forward completeness of the stochastic process. When SLFs non-differentiable at the origin can guarantee only transient-type

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properties of the origin, it is practically important to do another analysis for adding information about the existence of the process globally in time. It would be useful if one could use a single positive definite function to establish the global existence as well as transient stability properties at the same time.

Preliminary versions of materials presented in this paper were reported in Nishimura and Ito (2016) which contains brief sketches of some proofs. The present paper not only provides full proofs, but also improves the preliminary results by addressing the global existence of solutions even in the case of transient-type stability properties. Further, the material in Section 3 is reexamined and reorganized by strictly following the terminology employed in Itô (2004), Itô et al. (2012), and Øksendal (2013).

Notation. $\mathbb{R}^d$ is the d-dimensional Euclidean space, especially $\mathbb{R} := \mathbb{R}^1$. $|x|$ denotes the Euclidean norm of $x \in \mathbb{R}^n$. A continuous function $y : [0, \infty) \to [0, \infty)$ is said to be of class $\mathcal{K}$ if it is strictly increasing and $y(0) = 0$. A class $\mathcal{K}$ function y is said to be of $\mathcal{K}_\infty$ if $\lim_{s\to\infty} y(s) = \infty$. A continuous function $\mu : [0, \infty) \times [0, \infty) \to [0, \infty)$ is said to be of class $\mathcal{KL}$ if, for each fixed $t \in [0, \infty)$, the function $\mu(\cdot, t)$ is of class $\mathcal{K}$ and, for each fixed $s > 0$, $\mu(s, \cdot)$ is strictly decreasing and $\lim_{t\to\infty} \mu(s, t) = 0$. For $a, b \in \mathbb{R}$, let $a \wedge b$ denote the minimum of a and b. Let $\mathcal{P} := (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a filtered probability space, where Ω is the sample space, $\mathcal{F}$ is the σ-algebra that is a collection of all the events, $\{\mathcal{F}_t\}_{t\geq 0}$ is a filtration of $\mathcal{F}$, and $\mathbb{P}$ is a probability measure. The probability of some event A and the expectation of some random variable X are written as $\mathbb{P}[A]$ and $\mathbb{E}[X]$, respectively. The function $w := [w_1, w_2, \ldots, w_d]^T \in \mathbb{R}^d$ is a d-dimensional standard Wiener process defined on $\mathcal{P}$. The differential form of Stratonovich integral of a function $\sigma_\alpha : \mathbb{R}^n \to \mathbb{R}^n$ in $w_\alpha(t)$ is denoted by $\sigma_\alpha(x) \circ dw_\alpha(t)$ for $\alpha = 1, 2, \ldots, d$.

2. Motivating examples

Consider the stochastic differential equation of the form

$$dx(t) = -ax(t)dt + 2bx(t) \circ dw(t) \tag{1}$$

with $a, b > 0$. The origin $x = 0 \in \mathbb{R}$ is an equilibrium. Because the solution to (1) is $x(t) = x(0)\exp(-at + 2bw(t))$ as explained in Remark 5.5 of Khasminskii (2012), $x(t)$ converges to the origin with probability one. Picking $V_2(x) = x^2$, $(\mathcal{L}V_2)(x) = 2(-a + 4b^2)x^2 \geq 0$ holds[1] for $a \leq 4b^2$. Hence, $V_2(x) = x^2$ does not describe the convergence. For $V_p(x) = |x|^p$ with $p < a/(2b^2)$, one obtains

$$(\mathcal{L}V_p)(x) = p(-a + 2b^2 p)|x|^p < 0, \ \forall x \in \mathbb{R} \setminus \{0\}. \tag{2}$$

This suggests that the use of small p allows $V_p(x) = |x|^p$ to verify the convergence of $x(t)$ to the origin. However, employing small p makes V_p non-C^2 at $x = 0$, while the definition of $\mathcal{L}V(x)$ requires V to be C^2. It is interesting to know what kind of a conclusion $V(x)$ can draw actually about stability when it is not C^2 at the origin.

Allowing Lyapunov functions to be non-C^2 at the origins is not as simple as allowing non-differentiable functions for deterministic systems. Consider

$$dx(t) = -x(t)dt + 1 \circ dw(t), \tag{3}$$

where $x(t), w(t) \in \mathbb{R}$. The choice $V_1(x) = |x|$ yields

$$(\mathcal{L}V_1)(x) = -|x|, \quad \forall x \in \mathbb{R} \setminus \{0\}. \tag{4}$$

Hence, one may expect that the origin is stable and attractive in some sense. However, the origin is not an equilibrium because $x(0) = 0$ does not result in $x(t) \equiv 0$. Indeed, the solution to (3) with $x(0) = 0$ is

$$x(t) = e^{-t} \int_0^t e^s dw(s).$$

Thus, Itô isometry (e.g., Khasminskii, 2012; Mao, 2007; Øksendal, 2013) leads one to

$$\mathbb{E}[|x(t)|^2] = e^{-2t} \int_0^t e^{2s} ds = \frac{1}{2}(1 - e^{-2t}).$$

This yields $\mathbb{E}[|x(t)|^2] \to 1/2$ as $t \to \infty$. The variance of $x(t)$ stays away from zero and does not converge to zero, which seems to contradict the inference of stability and attractivity of $x = 0$ from (4). This indicates that a delicate issue comes into play when stochastic noises prevent the origin from being an actual equilibrium. The use of $V(x)$ which is not C^2 *at the origin* requires caution.

3. Properties of the origin

3.1. SDE and delicacy of the origin

This paper aims to analyze stability of the origin for the Stratonovich-type stochastic differential equation

$$dx(t) = f(x(t))dt + \sum_{\alpha=1}^{d} \sigma_\alpha(x(t)) \circ dw_\alpha(t), \tag{5}$$

where the drift term $f : \mathbb{R}^n \to \mathbb{R}^n$ is supposed to be locally Lipschitz, and the derivatives σ'_α of the diffusion coefficients $\sigma_\alpha : \mathbb{R}^n \to \mathbb{R}^n$ are assumed to be locally Lipschitz. The initial state is deterministic and denoted by $x(0) = x_0 \in \mathbb{R}^n$. For each $x_0 \in \mathbb{R}^n$, the local Lipschitz condition of f and σ'_α ensures the existence of a unique stochastic continuous solution locally in time.[2]

System (5) is time-invariant in the sense that f and σ_α are just functions of x, not t. Throughout this paper, $f(0) = 0$ is assumed. Hence, the origin is an equilibrium if and only if $\sigma_\alpha(0) = 0$ holds for all $\alpha = 1, 2, \ldots, d$. Taking into consideration the case of $\sigma_\alpha(0) \neq 0$, the origin $x = 0$ is only an "supposed" equilibrium of (5). Regardless of whether or not $\sigma_\alpha(0) = 0$ holds, this paper is interested in assessing stability of the origin $x = 0$.

3.2. Characteristic behavior of the origin

For deterministic systems, stability is usually defined as a property of equilibria. In contrast, as illustrated in Section 2, for stochastic systems, stability of non-equilibria can become of interest and it involves delicate issues. To investigate them, consider a stochastic process $x(t) \in \mathbb{R}^n$ as the solution to (5) with the initial condition $x(0) = 0 \in \mathbb{R}^n$. The origin $x = 0$ is not assumed to be an equilibrium. Define the following random variables:

$$e_{\neq} := \inf\{t > 0 \mid x(t) \neq 0\}, \tag{6}$$
$$e_0 := \inf\{t > 0 \mid \exists r > 0 \text{ s.t. } x(t \wedge \tau_r) = 0\}, \tag{7}$$

where $\tau_r \in [0, \infty]$ is the first exit time associated with $U_r := \{x \in \mathbb{R}^n \mid |x| < r\}$. The convention $\infty = \inf \emptyset$ is employed for (6) and (7). Note that the events $e_{\neq} = 0$ and $e_0 = 0$ are not mutually exclusive since there can be a lot of oscillatory sample paths. In other words, $\mathbb{P}[e_{\neq} = 0] = 1$ can hold true even in the case of $\mathbb{P}[e_0 = 0] = 1$. It is worth mentioning that for an

[1] $\mathcal{L}$ is the infinitesimal operator corresponding to the Lie derivative for deterministic systems.

[2] The existence of a sufficiently small time interval is guaranteed (see Khasminskii, 2012). The infinite interval $[0, \infty)$ is guaranteed if f and σ'_α are globally Lipschitz and satisfy the linear growth condition (see Khasminskii, 2012; Øksendal, 2013). Here, the globally Lipschitzness and the linear growth condition are not assumed. Instead, $[0, \infty)$ for the process $x(t)$ is focused and guaranteed in Section 6.