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The Aizerman and Kalman conjectures using symmetry☆

Ross Drummond *, Stephen Duncan

Department of Engineering Science, University of Oxford, 17 Parks Road, OX1 3PJ Oxford, United Kingdom

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ABSTRACT

Using a decomposition of a Lurie system in terms of symmetric and skew-symmetric matrices, this paper presents reformulations of the classical conjectures of Aizerman and Kalman which give valid conditions for absolute stability. Under this decomposition, it is shown that a restatement of the Aizerman conjecture implies stability while the re-stated Kalman conjecture implies contraction.

1. Introduction

The classical conjectures of Aizerman (1949) and Kalman (1957) propose Routh–Hurwitz type stability criteria for a class of nonlinear systems known as Lurie systems (Khalil, 2002). These conjectures have been shown to be false by counter-example (Barabanov, 1988; Fitts, 1966; Heath, Carrasco, & de la Sen, 2015), but in this paper, restatements of these conjectures are given such that they represent valid stability tests. The key result underpinning the restatements is the decomposition of the square matrices of the "closed-form" Lurie system into symmetric and skew-symmetric components. It is noted that the presented results are applicable to generic SISO Lurie systems and do not rely upon the system admitting a symmetric realisation.

2. Notation

The set of real matrices of dimension $n \times m$ is denoted by $\mathbb{R}^{n\times m}$, the set of symmetric matrices of dimension n as $\mathbb{S}^n$, the set of skew-symmetric matrices of dimension n as $\mathbb{A}^n$. The square matrix of dimension $n \times n$ containing 0s is denoted $\mathbf{0}^{n\times n}$. We adopt the notation of contraction theory from Forni and Sepulchre (2014). $\mathcal{M}$ is used to define a manifold of dimension n. $T_x\mathcal{M}$ denotes the tangent vector spaces at $x \in \mathcal{M}$ with $\delta x \in T_x\mathcal{M}$ a vector in the tangent space at x. The tangent bundle $T\mathcal{M}$ is the disjoint union of these tangent vector spaces over $\mathcal{M}$.

3. Lurie systems

The class of nonlinear systems considered by the Aizerman and Kalman conjectures are of the Lurie type.

Definition 1 (*Lurie System*)**.** The Lurie system is defined by the vector field

$$\begin{cases} \dot{x} &= Ax + B\phi(y) \\ y &= Cx \end{cases} \tag{1}$$

with $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times 1}$, $C \in \mathbb{R}^{1\times n}$ and $x \in \mathcal{M}$. We restrict our attention to the case of $\phi(y) : \mathcal{Y} \subseteq \mathbb{R} \to \mathbb{R}$ being a single nonlinear term that is assumed to be a C^2 function, Lipschitz on $\mathcal{Y}$ satisfying $\phi(0) = 0$, globally *sector bounded*

$$\frac{\phi(y)}{y} \in \Delta := [\underline{\mu}, \overline{\mu}] \quad \forall y \in \mathcal{Y} \subseteq \mathbb{R} \tag{2a}$$

and *slope restricted*

$$\frac{d\phi(y)}{dy} \in \Pi := [\underline{\pi}, \overline{\pi}] \quad \forall y \in \mathcal{Y} \subseteq \mathbb{R}. \tag{2b}$$

The following (false) conjectures of Aizerman (1949) and Kalman (1957) were proposed for verifying the stability of the Lurie system (1) in terms of Routh–Hurwitz type criteria. The form of the conjectures used here is in terms of the state-space matrices of the Lurie system.

[Aizerman Conjecture] If the matrix inequalities

$$A + kBC < 0 \quad \forall k \in \Delta \tag{3}$$

hold, then the nonlinear Lurie system is stable.

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* Corresponding author.
E-mail addresses: ross.drummond@eng.ox.ac.uk (R. Drummond), stephen.duncan@eng.ox.ac.uk (S. Duncan).

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[Kalman Conjecture] Similarly, if the matrix inequalities

$$A + kBC < 0 \quad \forall k \in \Pi \tag{4}$$

hold, then the Lurie system (1) is stable.

The Aizerman criterion was posed as a question in Aizerman (1949) while the Kalman conjecture was formulated as a criterion in Kalman (1957). Counter-examples have been found that disprove both conjectures in general (Bragin, Vagaitsev, Kuznetsov, & Leonov, 2011), with a comprehensive discussion of these conjectures given in Leonov and Kuznetsov (2013). For first and second order continuous time systems, the Aizerman conjecture is known to be true (Lozano, Brogliato, Egeland, & Maschke, 2013), but counter-examples exist for third order systems (Fitts, 1966). Similarly, third order counter-examples have been found for the Kalman conjecture (Barabanov, 1988) and a second order counter example was found for the discrete time case in Heath et al. (2015). The Aizerman conjecture was shown to be true when the linear system is described by a "strictly negative imaginary" transfer function (Carrasco & Heath, 2017) or when it has a positive impulse response (Gil, 1983). Even though the conjectures were shown to be false in general, valid techniques have been developed to verify stability of Lurie type systems, with this problem known as the absolute stability problem (Khalil, 2002).

4. Symmetrical decompositions of Lurie systems

This section introduces the symmetrical decomposition of the Lurie system matrices that is used in the re-statements of the Aizerman and Kalman conjectures. The essential result is that the closed-form vector field of the Lurie system can be decomposed into symmetrical and skew-symmetrical components. The proof of this result relies on the following basic property of square matrices.

Lemma 1. *Each square matrix $M \in \mathbb{R}^{n\times n}$ can be uniquely decomposed into the sum of a symmetrical matrix $M_s = \frac{1}{2}(M + M^T) \in \mathbb{S}^{n\times n}$ and a skew-symmetrical matrix $M_a = \frac{1}{2}(M - M^T) \in \mathbb{A}^{n\times n}$ such that $M = M_s + M_a$.*

Proof. The result is well known from linear algebra. See Horn and Johnson (2012) for instance. ∎

Define the *closed-form* Lurie system as

$$\dot{x} = Ax + \frac{\phi(y)}{y}BCx \tag{5}$$

where both A and BC are square matrices in $\mathbb{R}^{n\times n}$. This form is obtained by noting that $B\phi(y) = B\frac{\phi(y)}{y}y = \frac{\phi(y)}{y}BCx$ for the SISO case considered. Lemma 1 can then be used to uniquely decompose this vector field into symmetric and skew-symmetric components

$$\dot{x} = A_s x + A_a x + \frac{\phi(y)}{y}(BC)_s x + \frac{\phi(y)}{y}(BC)_a x \tag{6}$$

where $\{A_s, (BC)_s\} \in \mathbb{S}^{n\times n}$ and $\{A_a, (BC)_a\} \in \mathbb{A}^{n\times n}$ with the subscripts s and a respectively denoting the symmetric and skew-symmetric components of the square matrices.

For the re-statement of the stability conjectures, the following pair of matrices characterising the closed loop Lurie system dynamics of (5) are defined.

Definition 2 ($\Sigma,\ \Sigma_s,\ \Sigma_a$). Define the matrix pairs $\Sigma = \{A, BC\}$, $\Sigma_s = \{A_s, (BC)_s\},\ \Sigma_a = \{A_a, (BC)_a\}$.

5. Stability and the Aizerman conjecture

A re-statement of the Aizerman conjecture is now presented that provides a valid test for stability.

Theorem 1. *If the matrix Hurwitz condition of the Aizerman conjecture*

$$A + kBC < 0 \quad \forall k \in \Delta \tag{7}$$

holds when the matrix pair $\Sigma = \{A, BC\}$ are replaced by $\Sigma_s = \{A_s, (BC)_s\}$, then the origin is an asymptotically equilibrium point of the Lurie system.

Proof. The result is based upon the candidate Lyapunov function $V_A(x) = x^T x$ being the energy of the states of the Lurie system. The time domain derivative of $V_A(x)$ along the trajectories of (1) is $\dot{V}_A(x) = x^T\dot{x} + \dot{x}^T x$ where

$$\dot{V}_A = x^T\left(A + \frac{\phi(y)}{y}BC + \left(A + \frac{\phi(y)}{y}BC\right)^T\right)x \tag{8a}$$

$$= x^T\left(A_s + A_a + \frac{\phi(y)}{y}(BC)_s + \frac{\phi(y)}{y}(BC)_a\right. \tag{8b}$$

$$\left. + \left(A_s + A_a + \frac{\phi(y)}{y}(BC)_s + \frac{\phi(y)}{y}(BC)_a\right)^T\right)x \tag{8c}$$

$$= 2x^T\left(A_s + \frac{\phi(y)}{y}(BC)_s\right)x \tag{8d}$$

since $M_s + M_s^T = 2M_s$ and $M_a + M_a^T = \mathbf{0}^{n\times n}$ for $M_s \in \mathbb{S}^{n\times n}$ and $M_a \in \mathbb{A}^{n\times n}$. Then, if (3) holds for the matrix pair Σ_s, $\dot{V}_A < 0\ \forall\phi(y)/y \in \Delta$ and the level sets of $V_A(x)$ are positively invariant. ∎

6. Contraction and the Kalman conjecture

This section extends the asymptotic stability results of Section 5 by considering the convergence of the trajectories of the Lurie system towards each other with the conditions of the Kalman conjecture linked to contraction theory (Lohmiller & Slotine, 1998). A result of contraction theory says that if the symmetric part of the Jacobian of the systems dynamics is Hurwitz on some region, then the system trajectories are contracting and converge towards each other. This definition was applied in Zhang and Cui (2013) to Lurie systems.

The main concepts of contraction theory as given in Forni and Sepulchre (2014) are now described.

Definition 3 (*Contraction Metric Forni & Sepulchre, 2014*)**.** A contraction metric is defined as the positive definite function $F(x, \delta x) : \mathcal{M} \times T_x\mathcal{M} \to \mathbb{R}_{\geq 0}$ mapping the tangent bundle to the line of non-negative real numbers for every $(x, \delta x) \in \mathcal{TM}$.

Definition 4 (*Forward Invariant Set Forni & Sepulchre, 2014*)**.** The flow of the Lurie system from the initial condition $x_0 \in \mathcal{M}$ at time t_0 is denoted $\psi_{t_0}(\cdot, x_0)$. If $\psi_{t_0}(t, x_0) \in \mathcal{C}$ for each $t \geq t_0 \geq 0$, then the set $\mathcal{C} \subseteq \mathcal{M}$ is called forward invariant and connected. It is assumed that every two points in $\mathcal{C}$ can be connected by a smooth curve $\gamma : I \times \mathcal{C}$ with $I \triangleq \{s \in \mathbb{R} \mid 0 \leq s \leq 1\}$ that satisfies $\gamma(0) = x_1$, and $\gamma(1) = x_2$.

Definition 5 (*Findler Distance Forni & Sepulchre, 2014*)**.** Consider the contraction metric of Definition 3 defined on the manifold $\mathcal{M}$. For any two points $(x_1, x_2) \in \mathcal{C} \subseteq \mathcal{M}$, let $\Gamma(x_1, x_2)$ be the collection

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