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Stability of stochastic functional differential systems using degenerate Lyapunov functionals and applications[☆]

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ABSTRACT

Motivated by the seminal work of Dupire (2009) on functional Itô formulas, this work investigates asymptotic properties of systems represented by stochastic functional differential equations (SFDEs). Stability of general delay-dependent SFDEs is investigated using degenerate Lyapunov functionals, which are only positive semi-definite rather than positive definite as used in the classical work. This paper first establishes boundedness and regularity of SFDEs by using degenerate Lyapunov functionals. Then moment and almost sure exponential stabilities are obtained based on degenerate Lyapunov functionals and the semi-martingale convergence theorem. As an application of the stability criteria, consentability of stochastic multi-agent systems with nonlinear dynamics is studied.

1. Introduction

In his seminal work (Dupire, 2009), Dupire extended the Itô formula to a functional setting by using a pathwise functional derivative. This work has substantially eased the difficulties in finding solutions for non-Markovian processes due to time delays, that have defied bona fide operators and functional Itô formulas in the past. The current paper aims to use the newly developed functional Itô formula to examine stability and related issues of stochastic functional differential systems.

Many real systems, including population biology, epidemiology, economics, neural networks, and control of mechanical and electrical systems, inevitably involve delays, leading to delay differential equations (DDEs) and more general functional differential equations (FDEs) (Hale & Lunel, 1993; Kolmanovskii & Myshkis, 1999; Luo, Gong, & Jia, 2017). Some of their important properties have been established (Bejarano & Zheng, 2014; Bresch-Pietri & Krstic, 2014).

Although small delays may occasionally enhance stability in some systems (Yu, Chen, Cao, & Ren, 2013), delays are typically a source of instability, poor performance, and difficulty for analysis and control design. For linear time invariant systems, eigenvalues of system matrices may be used to deduce stability (Hale & Lunel, 1993). More complex functional equations are often treated by using Lyapunov functional and Razumikhin methods (Hale & Lunel, 1993; Kolmanovskii & Nosov, 1986). Combined with stochastic disturbances, delay systems become stochastic delay differential equations (SDDEs) and stochastic functional differential equations (SFDEs) (Mao, 1997; Mohammed, 1986), resulting in the consideration of moment stability, almost sure stability, and stability in probability. It is extremely difficult to establish necessary and sufficient conditions of mean square and almost sure stabilities for SFDEs. At present, Razumikhin methods and Lyapunov functionals are two main tools for establishing sufficient stability conditions. The stochastic versions of Razumikhin methods were developed in Janković, Randjelović, and Jovanović (2009), Mao (1996), Wu and Hu (2012) and Wu, Yin, and Wang (2015) for studying moment asymptotic stability. However, unlike its deterministic counterpart, the stochastic Razumikhin methods have limited success (Hale & Lunel, 1993; Teel, 1998).

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While sufficient conditions on delay-independent stability of SFDEs are often obtained by using Lyapunov functions, studies on delay-dependent stability mainly employ Lyapunov functionals and linear matrix inequalities (LMIs). Mao (1997) established the delay-independent pth-moment stability by using Lyapunov functions and obtained the almost sure stability from the moment exponential stability and a linear growth condition. Huang and Mao (2009a) studied the mean square exponential stability of neutral SDDEs by using a Lyapunov functional, and certain stability conditions were given in terms of LMIs. Caraballo, Real, and Taniguchi (2007) studied the almost sure exponential stability and ultimate boundedness of solutions for a class of neutral stochastic semi-linear partial delay differential equations. Rakkiyappan, Balasubramaniam, and Lakshmanan (2008) used Lyapunov functionals to derive LMI-type stability conditions for uncertain stochastic neural networks. Gershon, Shaked, and Berman (2007) investigated H_∞ state-feedback control of stochastic delay systems by Lyapunov functions and LMIs. Shaikhet (2013) introduced different Lyapunov functionals to examine stochastic stabilities of different SFDEs. These results were given under specific Lyapunov functionals. A fundamental stability theory related to Lyapunov functionals has not been developed.

Regarding delayed feedback stabilization problems of stochastic systems, Verriest and Florchinger (1995) gave delay-independent conditions on delayed feedback stabilization of linear SDDEs by Riccati-type equations in which multi-matrices must be determined. Huang and Mao (2009b) and Mao (2002) obtained delay-dependent exponential stability conditions for linear SDDEs in terms of LMIs. Karimi (2011) investigated robust mode-dependent delayed state feedback control for a class of uncertain delay systems in terms of LMIs.

All of the above references are based on positive definitive Lyapunov functionals. In this paper, using degenerate Lyapunov functionals, we develop several fundamental stability theorems, including moment asymptotic stability, exponential stability, and almost sure exponential stability. Degenerate Lyapunov functionals being positive semi-definite were well investigated in Kolmanovskii and Myshkis (1999) for deterministic neutral FDEs. Some specific degenerate Lyapunov functionals were also applied to examine the moment stability of stochastic functional differential equations with global Lipschitz conditions (Kolmanovskii & Myshkis, 1999; Kolmanovskii & Shaikhet, 1997; Shaikhet, 2013). Due to lack of bona fide operators and Itô formulas, whether degenerate Lyapunov functionals can be used to examine stability of SFDEs with nonlinear dynamics, nonlinear growth, and non-Lipschitz coefficients remains an open issue.

This paper fills in the gap. Our results reveal that appropriate degenerate Lyapunov functionals can be used to simplify stability analysis and control design. The results are then applied to multi-agent consentability and consensus problems, which are motivated by highway autonomous vehicles, unmanned aerial vehicles, team robots, among other applications (Cheng, Hou, & Tan, 2014; Li, Fu, Xie, & Zhang, 2011; Zong, Li, Yin, Wang, & Zhang, 2017; Zong, Li, & Zhang, 2017). While nonlinear multi-agent systems have attracted much attention (Strogatz, 2001; Zhu, Xie, Han, Meng, & Teo, 2017), stochastic multi-agent consentability and consensus with nonlinear dynamics, even for the delay-free case, are still not resolved. The existing LMI-type conditions are not explicit for consentability. The results of this paper resolve the issue. Assuming that the nonlinear terms satisfy a Lipschitz condition with uncertainty, we give sufficient stochastic consentability conditions, and demonstrate explicitly that the multi-agent consentability depends on the delay in feedback, noise intensities, balanced graphs, and the Lipschitz constant of the nonlinear dynamics.

The rest of the paper is structured as follows. Section 2 introduces the notation and develops the functional Itô formula for SFDEs. Section 3 establishes the fundamental degenerate Lyapunov functional theorem to produce the regularity and moment boundedness of SFDEs. Section 4 presents asymptotic moment estimates, moment stability, and almost sure stability based on the semi-martingale convergence theorem. In particular, mean square stability and almost sure stability are established for stochastic systems represented by semi-linear SFDEs. Section 5 applies the theoretical results to examine stochastic consentability of multi-agent systems with nonlinear dynamics. Section 6 concludes the paper with some remarks.

2. Functional Itô formula

We work with the n-dimensional Euclidean space $\mathbb{R}^n$ equipped with the Euclidean norm $|\cdot|$. For a vector or a matrix A, its transpose is denoted by A^T. For a matrix A, denote its trace norm by $|A| = \sqrt{\text{trace}(A^TA)}$. For a symmetric matrix A with real entries, denote by $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$ the largest and smallest eigenvalues, respectively. For two matrices A and B, $A \otimes B$ denotes their Kronecker product. Use $a \vee b$ to denote $\max\{a, b\}$ and $a \wedge b$ to denote $\min\{a, b\}$. For $\tau > 0$, denote by $C([-\tau, 0]; \mathbb{R}^n)$ the family of all $\mathbb{R}^n$-valued continuous functions on the interval $[-\tau, 0]$ with the norm $\|\varphi\|_C = \sup_{t\in[-\tau,0]}|\varphi(t)|$. Let $D([-\tau, 0]; \mathbb{R}^n)$ be the space of $\mathbb{R}^n$-valued functions on $[-\tau, 0]$ that are right continuous and have left-hand limits endowed with the Skorokhod topology. The metric in $D([-\tau, 0]; \mathbb{R}^n)$ is given by $d_D(x, y) = \inf\{\epsilon, \sup_{t\in[-\tau,0]}|t - \lambda(t)| < \varepsilon$ and $\sup_{t\in[-\tau,0]}|x(t) - y(\lambda(t))| < \varepsilon$ for some $\lambda(t) \in \Lambda\}$, where Λ is the space of the continuous increasing functions from $[-\tau, 0]$ onto $[-\tau, 0]$. Under this metric, D becomes a complete and separable metric space (Kushner, 1984). We work with $(\Omega, \mathfrak{F}, \mathbb{P})$, a complete probability space with a filtration $\{\mathfrak{F}_t\}_{t\geq 0}$ satisfying the usual conditions. Let $w(t) = (w_1(t), w_2(t), \ldots, w_d(t))^T$ be a d-dimensional standard Brownian motion. Consider the following system given by a stochastic functional differential equation (SFDE)

$$dy(t) = f(y_t, t)dt + g(y_t, t)dw(t), \tag{1}$$

where $y_t = \{y(t+\theta) : \theta \in [-\tau, 0]\}$, $f : C([-\tau, 0], \mathbb{R}^n) \times \mathbb{R}_+ \to \mathbb{R}^n$ and $g : C([-\tau, 0], \mathbb{R}^n) \times \mathbb{R}_+ \to \mathbb{R}^{n\times d}$ satisfy the following condition throughout the paper.

Assumption 2.1 (*Local Lipschitz Condition*). For each $j > 0$, there exists a constant $C_j > 0$ such that $|f(\psi, t) - f(\phi, t)| \vee |g(\psi, t) - g(\phi, t)| \leq C_j\|\psi - \phi\|_C$ for all $t \geq 0$ and $\psi, \phi \in C([-\tau, 0]; \mathbb{R}^n)$ with $\|\psi\|_C \vee \|\phi\|_C < j$.

We now develop the functional Itô formulation to the semi-martingale y_t determined by (1). For each $\varphi \in D([-\tau, 0]; \mathbb{R}^n)$, $\varphi_t(\theta) = \varphi(t + \theta), \theta \in [-\tau, 0]$. Motivated (Cont & Fournié, 2013; Dupire, 2009), we define the horizontal extension $\delta\varphi$ of φ: $\delta\varphi(\theta) = \varphi(\theta + \delta), \theta \in [-\tau, -\delta)$, and $\delta\varphi(\theta) = \varphi(0), \theta \in [-\delta, 0]$; and the vertical perturbation φ^v of φ: $\varphi^v(\theta) = \varphi(\theta), \theta \in [-\tau, 0)$, and $\varphi^v(0) = \varphi(0) + v, v \in \mathbb{R}^n$, Let $\{e_i, i = 1, \ldots, n\}$ denote the canonical basis in $\mathbb{R}^n$. Then for $\varphi \in D[-\tau, 0]$, we define

$$\mathcal{D}V(\varphi, t) := \lim_{\delta\to 0^+} \frac{V(\delta\varphi, t + \delta) - V(\varphi, t)}{\delta},$$

$\nabla_x V(\varphi, t) := \{\partial_i V(\varphi, t), i = 1, \ldots, n\}$,

$\partial_i V(\varphi, t) := \lim_{h\to 0} \frac{V(\varphi^{he_i}, t) - V(\varphi, t)}{h}$ and

$\nabla_{xx} V(\varphi, t) := \{\partial_{i,j} V(\varphi, t), i, j = 1, \ldots, n\}$,

$\partial_{i,j} V(\varphi, t) := \lim_{h\to 0} \frac{\partial_j V(\varphi^{he_i}, t) - \partial_j V(\varphi, t)}{h}$, where the limits are assumed to exist. We use $\mathcal{C}^{2,1}(D([-\tau, 0]; \mathbb{R}^n) \times \mathbb{R}_+; \mathbb{R})$ to denote the class of functions that have continuous derivatives up to the second order w.r.t. x and continuous derivative w.r.t. t.