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Optimal under-actuated kinematic motion planning on the ϵ-group ☆

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ABSTRACT

A global motion planning method is described based on the solution of minimum energy-type curves on the frame bundle of connected surfaces of arbitrary constant cross sectional curvature ϵ. Applying the geometric framing of Pontryagin's principle gives rise to necessary conditions for optimality in the form of a boundary value problem. This arbitrary dimensional boundary value problem is solved using a numerical shooting method derived from a general Lax pair solution. The paper then specializes to the 3-dimensional case where the Lax pair equations are integrable. A semi-analytic method for matching the boundary conditions is proposed by using the analytic form of the extremal solutions and a closed form solution for the exponential map. This semi-analytical approach has the advantage that an analytic description of the control accelerations can be derived and enables actuator constraints to be incorporated via time reparametrization. The method is applied to two examples in space mechanics: the attitude control of a spacecraft with two reaction wheels and the spacecraft docking problem.

1. Introduction

This paper addresses the motion planning problem on the frame bundle and isometry group of m-dimensional simply connected surfaces of constant cross sectional curvature ϵ. The motion planner considered here deals with the case where ϵ is arbitrary with frame bundle, coined here, the ϵ-group G_ϵ. While according to the Killing–Hopf theorem, any complete connected Riemann space M^m, $m \geq 2$ of constant curvature ϵ has a universal cover $\mathbb{S}^m$, $\mathbb{H}^m$ or $\mathbb{E}^m$ and so the value of ϵ is usually set to 1, 0 or -1, we consider $\epsilon \in [-1, 0) \cup (0, 1]$ so that $\epsilon = 0$ can be considered only as a limiting case. This is useful in that for $\epsilon \neq 0$ the trace form is non-degenerate and our optimal control problem reduces to solving an identity derived from a Lax pair form. Moreover, a simple iterative map can be developed from this identity to numerically solve for the optimal solution $g \in \mathsf{G}_\epsilon$. In the case $\epsilon = 0$ no such Lax pair form exists due to the fact that it has a degenerate bi-linear trace form, but its solution can be considered in the limit.

The structure-preserving numerical method for solving optimal trajectories in this paper can be applied to arbitrary dimensions and for arbitrary curvature of the underlying space form (approximately optimal in the case $\epsilon = 0$ where we approximate it by a very small number in part of the numerical integration). In addition, the paper specializes to the completely integrable 3-D case where the extremal curves can be solved explicitly in terms of elliptic functions. Furthermore, the exponential map used to solve the boundary conditions is solved in closed-form for arbitrary ϵ. This leads us to a novel semi-analytic formulation for solving this class of optimal control problems on 3-D Lie groups for arbitrary ϵ. Moreover, in the integrable case the velocities and acceleration components can be derived from the analytically defined extremal curves and as such time-parameterization can be used to ensure dynamic feasibility of the kinematically feasible solution.

As we consider m-dimensional Riemann spaces, the isometry group G_ϵ will correspond to the groups $\mathsf{SO}(m+1)$ (in the case $\epsilon = 1$), $\mathsf{SE}(m)$ (in the case $\epsilon = 0$) and $\mathsf{SO}(m, 1)$ (in the case $\epsilon = -1$). The dimension of G_ϵ is then $n = \frac{m(m+1)}{2}$. This paper initially considers systems whose configuration space $g \in \mathsf{G}_\epsilon$ satisfies the following differential constraint

$$\begin{cases} \dot{g} & = g(\sum_{i=1}^{n} v_i A_i) \\ g(0) & = g_0 \quad \text{and} \quad g(t_f) = g_d. \end{cases} \tag{1}$$

The vectors $[v_1, v_2, \ldots, v_n]^\top \in \mathbb{R}^n$ are continuous functions, $g \in \mathsf{G}_\epsilon$, and $A_1, \ldots, A_n$ is the basis of the ϵ-Lie algebra $\mathfrak{g}_\epsilon$.

There are a plethora of applications that can be modeled by (1). For example, the kinematics of various autonomous systems such as the attitude kinematics ($\epsilon = 1$, $n = 3$) of spacecraft (Biggs & Colley, 2016; Maclean, Pagnozzi, & Biggs, 2014; Spindler, 1996), including those with velocity constraints (Biggs & Horri, 2012) and those under-actuated in control (Biggs, Bai, & Henninger,

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2017; Spindler, 1999). Systems such as wheeled robots, robotic grippers and slender underwater vehicles that exhibit a sliding-type constraint ($\epsilon = 0, n = 2$ or $n = 3$) (Biggs & Holderbaum, 2009; Biggs, Holderbaum, & Jurdjevic, 2007; Bretl & McCarthy, 2014; Coverstone-Carroll, 1996; Dubins, 1957; Liu & Geng, 2014; Mukherjee, Emond, & Junkins, 1997; Murray & Sastry, 1993) and in wider fields such as switched electrical networks (Leonard & Krishnaprasad, 1994), problems of quantum control (D'Allessandro, 1993) and planning curvature- and torsion-constrained 3D printed implants for facilitating radiation therapy (Patil, Pan, Abbeel, & Goldberg, 2015). Various methods have been developed to tackle the motion planning problem of left-invariant (respectively right) systems defined on matrix Lie groups of the form (1) where it is necessary to match the boundary condition $g(0) = g_0$ and $g(T) = g_T$. For example, the works (Brockett & Dai, 1991; Murray & Sastry, 1993) introduce the idea of solving non-holonomic motion planning problems by expressing the control analytically in terms of either elliptic functions (Brockett & Dai, 1991) or sinusoids (Murray & Sastry, 1993). The parameters of these analytic control functions must then be computed to match the boundary conditions imposed on the motion planning problem. While the method we construct in this paper considers systems with non-holonomic constraints, for the purposes of controls these systems are treated as kinematic systems, i.e., the velocities are assumed to be directly controlled or equivalently the dynamics of the system can be perfectly canceled with the control. The distinction between these systems and dynamic non-holonomic systems is detailed in Bloch et al. (2003). In Spindler (1996), the author solves the necessary conditions for optimality using Pontryagin's principle and suggests using a standard numerical shooting method to solve for the boundary conditions via numerical integration. The paper (Leonard & Krishnaprasad, 1995) applies classical averaging theory; they produce sinusoidal controls that solve this motion planning problem with $O(\epsilon^p)$ accuracy in general, and exactly if the Lie algebra is nilpotent. The projection to the group is determined in local coordinates using the Wei–Norman product of exponentials representation and the Magnus single exponential representation. The paper (Chitour, 2002) solves the problem for semi-simple and compact Lie groups via a continuation method. The restriction to a compact group is crucial to their handling, since the continuation method requires that the Wazewski equation must have a global solution. Lafferriere and Sussmann (1991) propose a general strategy for solving (1)–(19) by making use of an extended system, which comprises the original system plus higher-order Lie brackets of the system vector fields. The control which is determined by such a system provides an exact solution of the original problem if the given system is nilpotent or for the class of systems they classify as "feedback nilpotentizable", and for all other systems the solutions are approximate. The thesis (Baillieul, 1975) analyzes the sub-Riemannian optimal control problems on $\mathsf{SO}(3)$ using a variational approach, while Brockett (1973) and Jurjdevic (1997) use the Pontryagin maximum principle. The work (Bloch, Crouch, & Ratiu, 1994) analyzes the Hamiltonian structure of kinematic optimal control problems, particularly the sub-Riemannian optimal control problems on compact semi-simple Lie groups and gives a Lax pair form defining the necessary conditions for optimality for the special case where G_ϵ is the frame bundle of a Riemannian symmetric space; in this paper, G_ϵ is semi-simple and compact only in the case $\epsilon > 0$.

The general approach in this paper is to focus on a class of optimal solutions to the motion planning problem. Moreover, in addition to considering matching the boundary conditions the following quadratic cost is imposed

$$\mathcal{J} = \int_0^{t_f} \sum_{i=1}^{s} c_i v_i(t)^2 dt \tag{2}$$

where $s \leq n$ the time t_f is a fixed variable and $c_i > 0$ are constant scalar weights. Integrability of the extremal equations for an optimal control problem of this type has been detailed in Bloch et al. (2003). Minimizing the cost (2) leads us to solve the system (1) as an optimal control problem, using the geometric framing of Pontryagin's maximum principle. However, the boundary conditions are not contained in the cost function and to match them specific values of the initial conditions of the extremal curves have to be computed. In this paper we derive an identity from the general Lax Pair solution that arises from this optimal control problem on Lie groups and use it to construct an iterative approach to solve the motion planning problem for prescribed boundary conditions. The approach has the advantages over previous methods as (i) it generalizes to a large class of n-dimensional systems with a left-invariant differential constraint defined on the frame bundles of spaces of arbitrary constant cross sectional curvature. (ii) It does not require any analytical approximation methods such as averaging. (iii) The curve $g(t)$ on the group is a global, co-ordinate-free solution. This means that the method avoids singularities and the un-winding problem that can be encountered when parameterizing the group. (iv) The derived numerical shooting and integration method used for matching the boundary conditions preserves the first integral and the structure of the group.

The last section of the paper specializes to the completely integrable 3-D case for arbitrary ϵ. For the case where the optimal control problem lifts to a quadratic Hamiltonian a general solution to the extremal curves are explicitly solved in terms of Jacobi elliptic functions. Furthermore, a closed-form solution of the exponential map is derived which allows the construction of a semi-analytical shooting method. Due to the semi-analytical nature of this case the required velocities, accelerations and controls can also be constructed analytically. The integrable cases, therefore, lend themselves to the possibility of time-parameterization which can be used to ensure dynamic feasibility in practical problems. To this end the method is applied to two problems in space mechanics (1) the slewing of an underactuated spacecraft using only two reaction wheels and (2) a spacecraft docking problem where the spacecraft can only thrust in the forward and backward directions of the body-fixed frame and it must rotate to point the thrusters in the required direction in inertial space.

2. Optimal trajectories of minimum-energy type on the ϵ-group

This section presents background to the geometric framing of Pontryagin's principle for the m-dimensional ϵ-group and some well-known properties of G_ϵ and its Lie algebra $\mathfrak{g}_\epsilon$ (Section 2.1) as well as the form of the Lax pair equations for the n-dimensional case. A novel identity is constructed expressing the extremal curves on the Lie algebra $\mathfrak{g}_\epsilon$ in terms of the curve $g(t)$ which holds for all ϵ values ($\epsilon = 0$ is approached as a limiting case as $\epsilon \to 0$). This identity is then used to construct a simple iterative method alongside a first-order Lie symplectic Euler scheme to determine the initial extremals that are required to match the boundary conditions on the group.

2.1. Background

Here we review some known facts of optimal control on matrix Lie groups and facts about the ϵ-Lie algebra and fix notation. References used are Bloch et al. (2003, 1994), Judjevic(2005), Jurdjevic (1997, 2001) and Jurdjevic and Sussmann (1972).

Given the matrix $g \in \mathsf{G}_\epsilon$, and defining the $m+1 \times m+1$ matrix $J_\epsilon = \mathrm{diag}(1, \ldots, 1, \frac{1}{\epsilon})$, then for all integer ϵ,

1. $gJ_\epsilon g^\top = J_\epsilon$
2. $\det(g) = 1$.