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Brief paper

Uniform convergence for signed networks under directed switching topologies[☆]

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ABSTRACT

This paper contributes to developing necessary convergence conditions for directed signed networks subject to cooperative and antagonistic interactions. A class of Laplacian-dependent convergence conditions is presented in the presence of switching topologies. It is shown that the switching signed networks converge monotonically to the intersection space of the null spaces of all Laplacian matrices. Furthermore, the uniform bipartite consensus (respectively, uniform asymptotic stability) of switching signed networks is tied closely to the simultaneous structural balance (respectively, unbalance) of the switching signed digraphs associated with them.

1. Introduction

Networks and coordination problems have received considerable research efforts in recent years, for which the graph theoretic approaches play a fundamentally important role in the convergence analysis. One of the well-studied classes of networks is described by the (graph) Laplacian-type flow, which aims at enabling all nodes to reach agreement on a common quantity of interest. This is usually achieved thanks to the cooperative efforts of all nodes, represented by a traditional graph with positive edge weights, which however can admit only the unified (i.e., cooperative) interactions among nodes. In contrast, there are circumstances that need to consider different interactions among nodes. A well-known example is the social networks, in which it may involve two basically different kinds of relationships among nodes, like friendly/hostile, like/dislike, and trust/distrust interactions. This class of examples admits the simultaneous existence of different cooperative and antagonistic interactions among nodes (see Altafini (2013a); Altafini and Lini (2015); Jiang, Zhang, and Chen (2017); Proskurnikov and Cao (2017); Valcher and Misra (2014); Yaghmaie, Su, Lewis, and Olaru (2017)).

Networks with antagonistic interactions have become one of the most attractive fields for automatic control, especially after the notion of bipartite consensus has been proposed recently in Altafini (2013a). We call this class of networks *signed networks*, in which the interactions among nodes are conveniently described by signed graphs with positive and negative weights (see Zaslavsky (1982)). As demonstrated in Altafini (2013a), signed networks include as a special case *traditional networks* whose interactions are expressed by traditional graphs. The Laplacian matrices of traditional and signed graphs have two common features, i.e., (1) diagonal dominance and (2) strict diagonal dominance in none of the coordinates. By considering how to well benefit from them, many remarkable convergence results have been given in the literature, e.g., see Cheng, Wang, and Hu (2008), Meng and Jia (2016), Ren and Beard (2005), Su and Huang (2011) and Zhang and Jia (2014) for traditional networks and Altafini (2013b), Hu & Zheng (2014), Liu, Chen, & Başar (2016), Liu, Chen, Başar, & Belabbas (2015, 2017), Meng, Du, & Jia (2016), Proskurnikov, Matveev, & Cao (2016), Shi, Proutiere, Johansson, Baras, & Johansson (2015) and Xia, Cao, & Johansson (2016) for signed networks.

Unlike traditional networks, signed networks are subject to challenging difficulties in the presence of switching topologies. A direct reason is that the convergence issues cannot be solved

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by theories for nonnegative (stochastic or substochastic) matrices and M-matrices and algebraic graph theories associated with them (see, e.g., Horn and Johnson (1985, 1991)). For a particular case when the signed digraphs are all structurally balanced and can be rendered simultaneously nonnegative by a unified gauge transformation, it has been discussed in Altafini (2013a) that the results for traditional switching networks are effectively extended to cope with switching signed networks (see Altafini (2013a, Subsection III-B-2, p. 941)). However, such discussions of Altafini (2013a) impose an assumption that each signed digraph in the family of digraphs is both digon sign-symmetric and strongly connected. It has been shown by contrast that the repeated joint strong connectivity is sufficient to achieve the modulus or bipartite consensus of switching signed networks represented by the Altafini's model in both discrete-time and continuous-time domains (see, e.g., Liu et al. (2016, 2015, in press); Xia et al. (2016)). Recently, a new approach using general differential inequalities instead of equalities has been proposed to model the Laplacian-type flow (algorithm or protocol) of networks in Proskurnikov and Cao (2017). Though the traditional graphs are used in the modeling, this inequality-based approach can be applied to cope with the convergence problems on signed networks, which in particular extends the results of Proskurnikov et al. (2016). It is worth pointing out, however, that the existing results of Altafini (2013a), Liu et al. (2016, 2015, 2017), Proskurnikov and Cao (2017), Proskurnikov et al. (2016) and Xia et al. (2016) focus only on providing conditions to ensure convergence of switching signed networks, where the properties of steady states have not been studied and the relationships between the structural balance and uniform convergence have not been fully explained.

In this paper, we focus on proposing conditions which are necessarily required by the uniform convergence of switching signed networks. Two classes of convergence conditions are developed. The first class of conditions depends on the Laplacian matrices of signed digraphs, which is necessary for switching signed networks to converge. The steady states for different cases are identified thoroughly. The second class of conditions exploits the simultaneous structural balance properties of switching signed digraphs. It is shown that if the switching signed networks achieve the uniform bipartite consensus (respectively, uniform asymptotic stability), then the switching signed digraphs associated with them are simultaneously structurally balanced (respectively, unbalanced). This discloses the relationship between simultaneous structure properties and uniform convergence performances of switching signed networks.

We organize the remainder of this paper as follows. In Section 2, we give convergence problems of interest for signed networks under switching topologies. The uniform convergence problems are addressed in Section 3, where Laplacian-dependent and structure-dependent convergence conditions are proposed. In Section 4, a brief conclusion is made. For clarity, the proofs of Lemma 1, Theorem 1 and Corollary 1 are provided in Appendices A–C, respectively.

Notations: Let $\mathscr{I}_n = \{1, 2, \ldots, n\}$ and $1_n = [1, 1, \ldots, 1]^{\mathrm{T}} \in \mathbb{R}^n$. For any $a \in \mathbb{R}$, we denote $|a|$ and $\operatorname{sign}(a)$ as the absolute value and sign value of a, respectively. We denote I_n as the nth-order identity matrix and 0 as the null matrix/vector with required dimensions, respectively. For any $A = [a_{ij}] \in \mathbb{R}^{m\times n}$, let us define $\mathscr{N}(A) = \{\xi \in \mathbb{R}^n : A\xi = 0\}$, $|A| = [|a_{ij}|]$, and $\Delta_A = \operatorname{diag}\{\sum_{j=1}^n a_{1j}, \sum_{j=1}^n a_{2j}, \ldots, \sum_{j=1}^n a_{mj}\} \in \mathbb{R}^{m\times m}$. We also denote the set of n-by-n gauge transformations as $\mathscr{D}_n = \{D = \operatorname{diag}\{d_1, d_2, \ldots, d_n\} : d_i \in \{\pm 1\}, i = 1, 2, \ldots, n\}$.

2. Convergence problems on signed networks

2.1. Signed digraphs

For a signed network, the interactions among nodes are conveniently described with a signed digraph (or directed graph) $\mathscr{G} = (\mathscr{V}, \mathscr{E}, \mathscr{A})$, in which $\mathscr{V} = \{v_i : i \in \mathscr{I}_n\}$, $\mathscr{E} \subset \mathscr{V} \times \mathscr{V} = \{(v_i, v_j) : v_i, v_j \in \mathscr{V}\}$ and $\mathscr{A} = [a_{ij}] \in \mathbb{R}^{n\times n}$ are a node set, an edge set, and a weighted adjacency matrix, respectively. We also have equivalent relations in $\mathscr{G}$ that for each $i, j \in \mathscr{I}_n$, $a_{ij} \neq 0 \Leftrightarrow (v_j, v_i) \in \mathscr{E}$ and $a_{ij} = 0$ otherwise. Let $(v_i, v_i) \notin \mathscr{E}$ or $a_{ii} = 0$, $\forall i \in \mathscr{I}_n$. If $(v_j, v_i) \in \mathscr{E}$ or $a_{ij} \neq 0$, $\forall j \neq i$, then $\mathscr{G}$ has an edge from v_j to v_i and v_j is a neighbor of v_i. Denote $\mathscr{N}_i = \{j : (v_j, v_i) \in \mathscr{E}\}$ as the label set of all neighbors of v_i. If $\mathscr{G}$ admits paths between every distinct pair of nodes, then it is said to be strongly connected.

A time-varying signed digraph is represented by $\mathscr{G}(t) = (\mathscr{V}, \mathscr{E}(t), \mathscr{A}(t))$ with $\mathscr{A}(t) = [a_{ij}(t)] \in \mathbb{R}^{n\times n}$, where $a_{ij}(t) \neq 0 \Leftrightarrow (v_j, v_i) \in \mathscr{E}(t)$ and $a_{ij}(t) = 0$ otherwise for all $j \neq i$ and $a_{ii}(t) = 0$, $\forall i \in \mathscr{I}_n$. Hence, v_i has a time-varying neighbors' label set, denoted by $\mathscr{N}_i(t) = \{j : (v_j, v_i) \in \mathscr{E}(t)\}$. We contribute to the nontrivial case $\mathscr{A}(t) \not\equiv 0$, $\forall t \geq t_0$ in the sequel for any initial time $t_0 \geq 0$.

We present two notions for structural balance and structural unbalance of the time-varying signed digraph $\mathscr{G}(t)$, $\forall t \geq t_0$.

Definition 1. A time-varying signed digraph $\mathscr{G}(t)$ is said to be simultaneously structurally balanced (s.s.b.) if there exists a time-invariant bipartition $\{\mathscr{V}^{(1)}, \mathscr{V}^{(2)} : \mathscr{V}^{(1)} \cup \mathscr{V}^{(2)} = \mathscr{V}, \mathscr{V}^{(1)} \cap \mathscr{V}^{(2)} = \emptyset\}$ of $\mathscr{V}$ such that $a_{ij}(t) \geq 0$, $\forall v_i, v_j \in \mathscr{V}^{(l)}$ $(l \in \{1, 2\})$, $\forall t \geq t_0$ and $a_{ij}(t) \leq 0$, $\forall v_i \in \mathscr{V}^{(l)}$, $\forall v_j \in \mathscr{V}^{(q)}$ $(l \neq q, l, q \in \{1, 2\})$, $\forall t \geq t_0$; and it is said to be simultaneously structurally unbalanced (s.s.ub.), otherwise.

From Definition 1, $\mathscr{G}(t)$ is s.s.b. if and only if there exists a constant $D \in \mathscr{D}_n$ to ensure $D\mathscr{A}(t)D = |\mathscr{A}(t)|$ for all $t \geq t_0$; and it is s.s.ub., otherwise. It is clear that Definition 1 shows further notions of structural balance and structural unbalance for signed graphs introduced in, e.g., Altafini (2013a, Definition 2).

2.2. Network dynamics

Consider a signed network associated with the time-varying signed digraph $\mathscr{G}(t)$, and the dynamics of each node satisfy

$$\dot{x}_i(t) = \sum_{j\in\mathscr{N}_i(t)} a_{ij}(t)\left[x_j(t) - \operatorname{sign}(a_{ij}(t))x_i(t)\right], \quad i \in \mathscr{I}_n \tag{1}$$

where $x_i(t) \in \mathbb{R}$ represents the information state of the node v_i. Denote $x(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^{\mathrm{T}}$, and hence the dynamics of (1) can be rewritten in a compact vector form of

$$\dot{x}(t) = -L(t)x(t) \tag{2}$$

where $L(t)$ is the Laplacian matrix of $\mathscr{G}(t)$, defined by

$$L(t) = [l_{ij}(t)] \in \mathbb{R}^{n\times n} \text{ with } l_{ij}(t) = \begin{cases} \sum_{k\in\mathscr{N}_i(t)} |a_{ik}(t)|, & j = i \\ -a_{ij}(t), & j \neq i. \end{cases} \tag{3}$$

For the system (2), we consider any initial time $t_0 \geq 0$ and denote $x(t_0) \triangleq x_0$ as any initial state. We also denote $\Phi(t, t_0)$, $\forall t \geq t_0$ as the state transition matrix of the system (2). When there are no confusions, these will be not indicated.

We are specifically interested in (2) for switching signed networks. Towards this end, let $\widehat{\mathscr{G}_\sigma} = \{\mathscr{G}_{\sigma_1}, \mathscr{G}_{\sigma_2}, \ldots, \mathscr{G}_{\sigma_M}\}$ of M finite elements denote all signed digraphs for switching signed networks that are represented by $\mathscr{G}(t)$, i.e., $\mathscr{G}(t) \in \widehat{\mathscr{G}_\sigma}$, $\forall t \geq t_0$, where $\mathscr{G}_{\sigma_p} = (\mathscr{V}, \mathscr{E}_{\sigma_p}, \mathscr{A}_{\sigma_p})$ with $\mathscr{A}_{\sigma_p} = [a_{ij,\sigma_p}] \in \mathbb{R}^{n\times n}$, $\forall p \in \mathscr{I}_M$. For any $\mathscr{G}_{\sigma_p}$, $\forall p \in \mathscr{I}_M$, we denote $\mathscr{N}_{i,\sigma_p}$ of v_i and L_{σ_p} similarly as $\mathscr{N}_i(t)$ of