Automatica 90 (2018) 16–24

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Brief paper

Design of measurement difference autocovariance method for estimation of process and measurement noise covariances☆

Jindřich Duník *, Oliver Kost, Ondřej Straka

Department of Cybernetics and European Centre of Excellence NTIS, Faculty of Applied Sciences, University of West Bohemia, Univerzitní 8, 306 14 Pilsen, Czech Republic

ARTICLE INFO

Article history:
Received 9 December 2016
Received in revised form 13 September 2017
Accepted 24 November 2017

Keywords:
Identification
State–space models
State estimation
Prediction
Noise covariance matrices

ABSTRACT

The paper deals with the estimation of the process and measurement noise covariance matrices of a system described by the linear time-varying state–space model. In particular, the stress is laid on the correlation methods and a novel method, the measurement difference autocovariance method, is designed. The proposed method is based on the statistical analysis of an augmented measurement prediction error leading to a system of linear matrix equations for the elements of the noise covariance matrices. Compared to other correlation methods, the proposed method provides unbiased estimates even for a finite number of measurements. The theoretical results are discussed and illustrated in a numerical example.

1. Introduction

Knowledge of a system model is a key prerequisite for many *optimal* state estimation, signal processing, fault detection, and optimal control algorithms. The model is often designed to be consistent with random behaviour of the system quantities and properties of the measurements. While the deterministic part of the model often arises from the first principles based on physical, chemical, or biological laws governing the behaviour of the system, the statistics of the stochastic part are often difficult to find by the modelling and have to be identified using the measured data. An incorrect description of the noise statistics may result in a significant worsening of estimation, signal processing, detection, or control quality or even in a failure of the underlying algorithms.

In the last five decades, therefore, a significant research interest has been focused on a design of the methods for estimation of the properties of the stochastic part of the model. The attention has been devoted to both the input–output models (Söderström & Stoica, 1989) and the state–space (SS) models (Bélanger, 1974; Kashyap, 1970; Lainiotis, 1971; Mehra, 1970; Odelson, Rajamani, & Rawlings, 2006; Shumway & Stoffer, 2000; Wiberg, Powell, & Ljungquist, 2000; Zhou & Luecke, 1995). This paper focuses on the methods estimating the properties of the stochastic part of the system described by a discrete-time SS model. In particular, the methods estimating covariance matrices[1] (CMs) of the noises in the state and measurement equation from a sequence of measured data are of interest. The methods are further denoted as the *noise CM estimation methods*.

In the literature, a large number of various noise CM estimation methods can be found. The methods differ in the assumptions related to the considered model, underlying ideas and principles, properties of the estimates, and number and essence of the design parameters. Traditionally, the noise CM estimation methods are divided into four groups (Duník, Straka, Kost, & Havlík, 2017; Mehra, 1972):

- *Correlation methods*, where the innovation sequence of a linear estimator, which is not optimal in the mean square error (MSE) sense, is statistically analysed (Bélanger, 1974; Duník, Straka, & Šimandl, 2017; Friedland, 1982; Kashyap, 1970; Lee, 1980; Lima & Rawlings, 2011; Mehra, 1970; Odelson et al., 2006; Šimandl & Duník, 2011; Zhou & Luecke, 1995).

☆ This work was supported by the Czech Science Foundation (GACR), grant GA 16-19999J, and the Czech Ministry of Education, Youth and Sports, project LO1506. The material in this paper was not presented at any conference. This paper was recommended for publication in revised form by Associate Editor Er-Wei Bai under the direction of Editor Torsten Söderström.

* Corresponding author.
E-mail addresses: dunikj@kky.zcu.cz (J. Duník), kost@kky.zcu.cz (O. Kost), straka30@kky.zcu.cz (O. Straka).

[1] In general, for a complete description of the noise the probability density function is required. Nevertheless, for many signal processing and decision-making methods knowledge of the first two moments of the noise, i.e., the mean and the CM, is sufficient. Often, the noises are assumed zero-mean and the problem of finding a description of the noises reduces to an estimation of the noise CMs.

https://doi.org/10.1016/j.automatica.2017.12.040
0005-1098/

- *Maximum likelihood methods*, which are based on a batch joint estimation of the state process and noise CMs elements by the maximisation of a likelihood function often utilising the expectation–maximisation (EM) algorithm (Bavdekar, Deshpande, & Patwardhan, 2011; Shumway & Stoffer, 2000).
- *Covariance matching methods*, where the estimate error CMs computed by a filter are made consistent with the actual state and measurement estimation error statistics (Myers & Tapley, 1976; Wang, Liu, Fan, & Zhang, 2016).
- *Bayesian methods*, which are based on a recursive joint estimation of the process (i.e., state) together with the noise CMs elements by a nonlinear state estimator (Lainiotis, 1971; Özkan, Šmídl, Saha, Lundquist, & Gustafsson, 2013; Wiberg & DeWolf, 1993; Wiberg et al., 2000).

A characterisation of the methods with their assumptions, properties, and limitations can be found in e.g., Duník, Šimandl, and Straka (2009), Duník, Straka, Kost et al. (2017), Maybeck (1982), Mehra (1972), Moghaddamjoo and Kirlin (1993), Odelson et al. (2006) and Vil-Valls, Closas, and Fernández-Prades (2015).

Among the groups, the *correlation methods* have attracted quite *considerable* attention in the past as they may provide unbiased estimates of the noise CMs with acceptable computational requirements even for high-dimensional systems without a requirement on distribution of the noises. The methods are based on a statistical analysis of the one-step or multi-step measurement prediction error. The methods have been pioneered for linear models by Mehra and Bélanger in Bélanger (1974), Mehra (1970) and further advanced in Åkesson, Jørgensen, Poulsen, and Jørgensen (2008), Lima, Rajamani, Soderstrom, and Rawlings (2013), Odelson et al. (2006), Rajamani and Rawlings (2009) and Šimandl and Duník (2011). The correlation methods have been proven to provide:

- *Asymptotically*[2] *unbiased* estimates for linear time-invariant (LTI) (Kashyap, 1970; Mehra, 1970; Odelson et al., 2006) and linear time-varying (LTV) models (Bélanger, 1974; Duník, Straka, Šimandl, 2017),
- Unbiased estimates for *LTI models* (Lee, 1980; Zhou & Luecke, 1995), and
- Unbiased estimates for limited class of LTV models where the *measurement matrix of the model is of the rank equal to the state vector dimension* (Duník, Straka, & Kost, 2016).

In this paper, a novel method, the *measurement difference autocovariance least-squares-based (MDA) method*, for estimation of the SS model process and measurement noise CMs is developed. The MDA method, which belongs to the correlation methods, is proven to provide *unbiased* estimates of the noise CMs of a *general* LTV model even for a finite number of measurements. The properties of the MDA method estimate are thoroughly discussed, analysed, and compared with other methods. The performance of the MDA method is illustrated in a numerical simulation.

The rest of the paper is organised as follows. In Section 2, a system description and noise CM matrix estimation by the correlation methods are introduced and the goal of the paper is particularised. The MDA method for LTV models is proposed and thoroughly analysed in Sections 3 and 4, respectively. The numerical illustration is provided in Section 5 and concluding remarks are drawn in Section 6.

[2] Asymptotically unbiased estimate becomes unbiased if and only if the number of data used for the estimate computation approaches infinity. For a finite set of data, the estimate is *biased*.

2. System definition and problem formulation

Let the following SS model of an LTV discrete-time dynamic stochastic system with additive noises

$$\mathbf{x}_{k+1} = \mathbf{F}_k\mathbf{x}_k + \mathbf{G}_k\mathbf{u}_k + \mathbf{w}_k, \quad k = 0, 1, 2, \ldots, \tau, \tag{1}$$

$$\mathbf{z}_k = \mathbf{H}_k\mathbf{x}_k + \mathbf{v}_k, \quad k = 0, 1, 2, \ldots, \tau, \tag{2}$$

be considered, where the vectors $\mathbf{x}_k \in \mathbb{R}^{n_x}$, $\mathbf{u}_k \in \mathbb{R}^{n_u}$, and $\mathbf{z}_k \in \mathbb{R}^{n_z}$ represent the unknown state of the system, known input, and the measurement at time instant k, respectively. The matrices $\mathbf{F}_k \in \mathbb{R}^{n_x\times n_x}$, $\mathbf{G}_k \in \mathbb{R}^{n_x\times n_u}$, and $\mathbf{H}_k \in \mathbb{R}^{n_z\times n_x}$ are known. The variables $\mathbf{w}_k \in \mathbb{R}^{n_x}$ and $\mathbf{v}_k \in \mathbb{R}^{n_z}$ are the process and measurement zero-mean white noises with *unknown* noise CMs $\mathbf{Q} \in \mathbb{R}^{n_x\times n_x}$ and $\mathbf{R} \in \mathbb{R}^{n_z\times n_z}$, respectively. The state and measurement noise sequences $\{\mathbf{w}_k\}$ and $\{\mathbf{v}_k\}$ are assumed to be mutually independent. The moments of the initial state are supposed to be *unknown*.

Assumption 1. The matrices $\mathbf{F}_k$, $\mathbf{G}_k$, $\mathbf{H}_k$, $\mathbf{Q}_k$, and $\mathbf{R}_k$ are assumed to be bounded $\forall k$, i.e., each element of the matrices is assumed to be *finite*. Further, the initial state is assumed to be bounded as well, i.e., $\|\mathbf{x}_0\| < \infty$, where $\|\cdot\|$ is a vector norm.

The aim of the noise CM estimation methods is to estimate the *unknown* noise CMs $\mathbf{Q}$ and $\mathbf{R}$ using the *available* input and output data and the *known* matrices $\mathbf{F}_k$, $\mathbf{G}_k$, and $\mathbf{H}_k$.

2.1. Noise covariance matrices estimation by correlation methods

The correlation methods have been continuously theoretically developed, analysed, and applied over past several decades (Bélanger, 1974; Duník, Straka, Šimandl, 2017; Mehra, 1970; Odelson et al., 2006; Rajamani & Rawlings, 2009). The methods are based on a decomposition of the principally nonlinear estimation task into three successive linear estimation operations:

i. Prediction of a measurement and computation of respective measurement prediction error (MPE).
ii. Estimation of the MPE (cross-)covariance matrices defining the autocovariance function of the MPE.
iii. Estimation of the noise CMs by the least-squares (LS) method on the basis of the MPE statistics.

The methods can be classified into two groups according to the measurement prediction computation; first, the methods with an *explicit* computation of the state prediction (Åkesson et al., 2008; Bélanger, 1974; Dee, Cohn, Dachler, & Ghil, 1985; Duník, Straka, Šimandl, 2017; Godbole, 1974; Lima et al., 2013; Lima & Rawlings, 2011; Mehra, 1970; Odelson et al., 2006), second, the methods with an *implicit* computation of the state prediction (Duník et al., 2016; Feng, Fu, Ma, Xia, & Wang, 2014; Kashyap, 1970; Lee, 1980; Zhou & Luecke, 1995).

The former group requires a design of a linear stable state predictor, which is typically not optimal[3] in the MSE sense. Its output is subsequently used for the measurement prediction computation. The predictor is realised by a recursive algorithm, which has to be initialised. Assuming an unknown initial condition of the system, the effect of the user-defined initial condition of the predictor subsides with $\tau \to \infty$ and the moments of the MPE reach the statistical steady-state. For this reason, the estimates of the noise CMs of the linear models are "only" *asymptotically unbiased*, i.e., unbiased as the number of measured data τ approaches infinity. These methods have been proposed for LTI models in Dee et al. (1985), Godbole (1974), Mehra (1970), Odelson et al. (2006)

[3] As the noise CMs are unknown, the optimum (Kalman) gain cannot be computed. The gain of the state predictor is considered as a user-defined parameter which leads to a sub-optimal predictor.