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An interactive tool to introduce the waterbed effect

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Abstract: Linear control systems are subject to design constraints. Most of these constraints are intrinsic properties of linear systems which can be explained in terms of complex variable theory. In this work a tool designed to introduce the waterbed effect. This tool offers different and simultaneous graphical representations of this effect. All these representations are interactive.

Keywords: Linear system limitations, waterbed effect, education in control

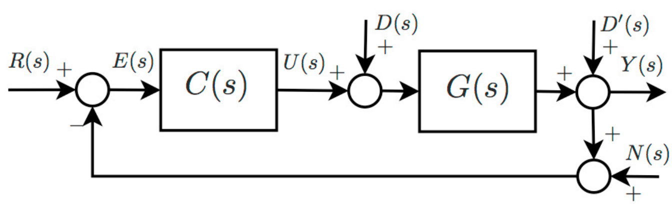

Figure 1. Studied system block scheme

1. INTRODUCTION

Interactive applications are teaching tools which can be used to introduce difficult concepts. Between others automatic control has been very active in this field (Dormido, 2004). During the last years a couple of courses based on this type of tools have appeared (Piguet and Longchamp, 2006; Longchamp, 2006) (Guzmán Sánchez et al., 2012), in both cases a set of interactive applications which cover all the course contents have been developed. Additionally, in (Guzmán Sánchez et al., 2012) a complete set of exercises to be performed with each app has been proposed.

Currently the authors are working to develop a set of applications which can be used in robust control courses (Skogestad and Postlethwaite, 2005; Doyle et al., 1992). In this work an application to describe the waterbed concept is presented.

The paper is organized as follows, section 2 introduces the standard control scheme and some of its characteristics, in section 3 contains the waterbed formulation and its proof, in section 4 develop application is described, in section 5 some waterbed examples are shown and in section 6 some conclusions and future works are discussed.

2. STANDARD CONTROL SCHEME

Figure 1 shows the standard closed-loop scheme, in it two main systems exist; the plant ($G(s)$) which corresponds to the system to be controlled and the controller ($C(s)$) which is the system to impose the desired behavior. Additionally, the closed-loop scheme contains several type of signals:

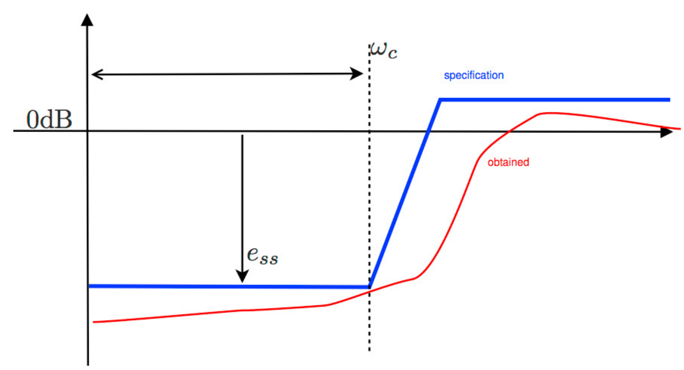

Figure 2. Standard specifications for $S(s)$.

- The output ($Y(s)$) is the variable to be regulated.
- The reference ($R(s)$) is the signal which is used to define the evolution of $Y(s)$.
- The disturbances ($D(s), D'(s)$) are external signals which effect should be compensated by the controller.
- The noise ($N(s)$) is a signal which can not be compensated but it should not be amplified.
- The error ($E(s)$) is a signal used to measure the control-loop performance.
- The control action ($U(s)$) which is the controller output.

The relationship between all these signals can be obtained through the following closed-loop transfer functions:

$$\begin{bmatrix} Y(s) \\ E(s) \\ U(s) \end{bmatrix} = \begin{bmatrix} T(s) & G(s)S(s) & S(s) & -T(s) \\ S(s) & G(s)S(s) & -S(s) & -S(s) \\ C(s)S(s) & T(s) & -C(s)S(s) & -C(s)S(s) \end{bmatrix} \begin{bmatrix} R(s) \\ D(s) \\ D'(s) \\ N(s) \end{bmatrix} \quad (1)$$

where $S(s) = \frac{1}{1+C(s)G(s)}$ is named sensitivity function (Doyle et al., 1992) and $T(s) = \frac{C(s)G(s)}{1+C(s)G(s)}$ is named complementary sensitivity function. Many control techniques are based on shaping the frequency response of these transfer functions. This can be done directly using H_∞ (Doyle et al., 1992) or μ-synthesis (Sánchez-Peña and Sznaier, 1998) methods or indirectly using loop-shaping methods

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over the open-loop transfer function $L(s) = C(s)G(s)$ (Sánchez-Peña and Sznaier, 1998).

As it can be shown in equation (1) most relevant input-output are closely related, so they cannot be independently shaped. As an example, it is well know that (Doyle et al., 1992):

$$S(s) + T(s) = \frac{1}{1 + C(s)G(s)} + \frac{C(s)G(s)}{1 + C(s)G(s)} = 1.$$

Similar relationships can be established between other transfer functions. In order to deal with these constraints a frequency band decomposition is usually established. Other important constraints exist in linear systems (Seron, 2010), between others, most relevant ones are phase-gain relationship, bandwidth limitations and the waterbed effect.

Given the closed-loop system introduced in Figure 1 and assuming that $G(s) = \frac{N_G(s)}{D_G(s)}$ and $C(s) = \frac{N_C(s)}{D_C(s)}$, with $N_G(s)$,$D_G(s)$ $N_C(s)$ and $D_C(s)$ being polynomials, the sensitivity function can be written as:

$$S(s) = \frac{1}{1 + C(s)G(s)} = \frac{D_C(s)D_G(s)}{N_C(s)N_G(s) + D_C(s)D_G(s)}$$

consequently the zeros of $S(s)$ are the open-loop poles (i.e. poles of $L(s)$). Additionally it can be stated that $S(s)$ is relative degree 0.

Usually, the sensitivity function is used to define the performance and it is the one used in shaping methods. Figure 2 shows, in blue, a standard specification for $|S(j\omega)|$ (also a possible solution is shown). Usually small values for $|S(j\omega)|$ are desired in the low-frequency range, this implies reduced error in the steady-state and in the low-frequency range. Also the bandwidth (ω_c) and the maximum amplification in the high-frequency range are design parameters.

The sensitivity function also offers a very interesting robustness measure, the minimal distance from $L(j\omega)$ to the point $-1 + j0$ in the Nyquist plot can be computed as:

$$\begin{aligned} \mathrm{d}\left(-1, L\left(j\omega\right)\right) &= \inf_{\omega} \left| -1 - L\left(j\omega\right) \right| = \inf_{\omega} \left|1 + L\left(j\omega\right)\right| \\ &= \left[\sup_{\omega} \frac{1}{\left|1 + L\left(j\omega\right)\right|}\right]^{-1} = \|S(s)\|_{\infty}^{-1}. \end{aligned}$$

3. THE WATERBED EFFECT

The developments in this section are based on the formulation shown in (Lewis, 2004; Astrom and Murray, 2012). In order to further analyze $S(s)$, the contour, Γ, shown in Figure 3 is defined. It is assumed that $S(s)$ is stable, and has n_{nmp} nonminimum-phase zeros, placed in $p_k \in \mathbb{C}^{+}$ [1]. Consequently $\ln\left(S(s)\right)$ is analytic over the contour shown in Figure 3 and:

$$\oint_{\Gamma} \ln\left(S(s)\right) ds = 0.$$

This contour integral can be decomposed in different components:

[1] Note that the zeros of $S(s)$ are the poles of $L(s)$, consequently nonminimum-phase zeros correspond to open-loop unstable poles.

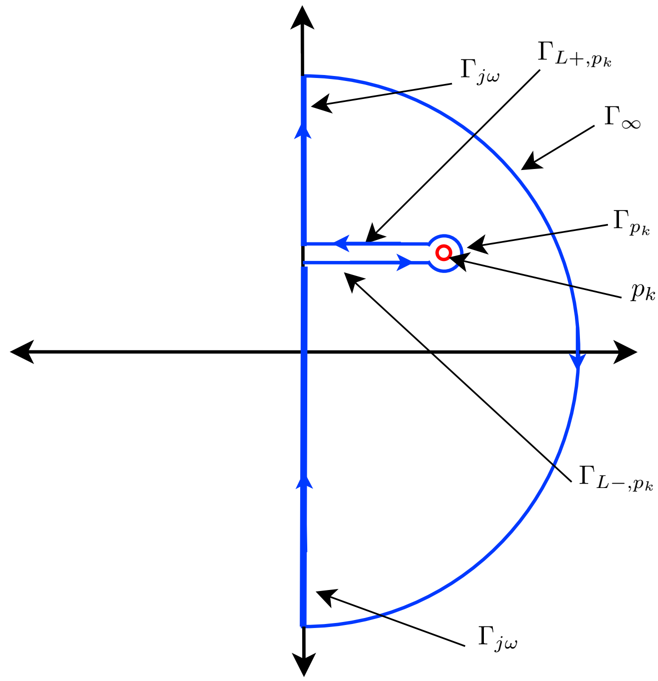

Figure 3. Waterbed contour (Γ).

$$\begin{aligned} \oint_{\Gamma} \ln\left(S(s)\right) ds &= \oint_{\Gamma_{j\omega}} \ln\left(S(s)\right) ds + \oint_{\Gamma_{\infty}} \ln\left(S(s)\right) ds \\ &+ \sum_{k=1}^{n_{nmp}} \left(\oint_{\Gamma_{p_k}} \ln\left(S(s)\right) ds + \oint_{\Gamma_{L+,p_k}} \ln\left(S(s)\right) ds \right) \\ &+ \sum_{k=1}^{n_{nmp}} \oint_{\Gamma_{L-,p_k}} \ln\left(S(s)\right) ds. \end{aligned}$$

In the following, this properties will be used:

- $\ln S(s) = -\ln(1 + L(s))$
- $\ln F(s) = \ln|F(s)| + j\angle F(s)$.
- $\int_{\Gamma} F(s)ds = \int_{\theta_0}^{\theta_1} F(\gamma\left(\theta\right)) \frac{\partial \gamma}{\partial \theta}\left(\theta\right) d\theta$, where $\gamma(\theta)$ is a parametrization of Γ.

In the following each component will be analyzed in detail:

- $\Gamma_{j\omega}$: This term corresponds to the frequency response integral.

$$\begin{aligned} \oint_{\Gamma_{j\omega}} \ln\left(S(s)\right) ds &= \int_{-\infty}^{\infty} j \ln\left(S(j\omega)\right) d\omega \\ &= j \int_{-\omega}^{\omega} \ln\left(|S(j\omega)|\right) d\omega - \int_{-\omega}^{\omega} \angle S(j\omega) d\omega \\ &= 2j \int_{0}^{\infty} \ln\left(|S(j\omega)|\right) d\omega \end{aligned}$$

- Γ_{∞} : This term corresponds to the contour integral around the semicircle of infinity radius ($R \to \infty$).

$$\oint_{\Gamma_{j\infty}} \ln\left(S(s)\right) ds = \int_{\frac{\pi}{2}}^{-\frac{\pi}{2}} \ln\left(S(R \cdot e^{j\theta})\right) j \cdot R \cdot e^{j\theta} d\theta$$

The term inside the ln when $R \to \infty$:

$$\lim_{R \to \infty} S(R \cdot e^{j\theta}) = \lim_{R \to \infty} \frac{1}{1 + L(R \cdot e^{j\theta})} = \frac{1}{1 + L(\infty) R^{-n_r} e^{-n_r \theta \cdot j}}$$

where $L(\infty) = \lim_{s \to \infty} L(s)$ and n_r is the relative degree of $L(s)$.