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Redefined observability matrix for Boolean networks and distinguishable partitions of state space☆

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ABSTRACT

This paper redefines the observability matrix and defines the observability index for Boolean networks (BNs). In the new definition, a BN is observable if and only if the observability matrix is of full rank. The observability matrix is calculated by a simple algorithm and is applied to the problem of distinguishability of partitions. A partition of the state space is said to be distinguishable if any two distinct subsets are distinguishable, and the finest distinguishable partition (FDP) is finer than any other distinguishable partition. A necessary and sufficient condition is proposed for checking the distinguishability of any given partition, and an algorithm is proposed for calculating the FDP. The proposed results are illustrated by examples.

1. Introduction

Kauffman proposed the Boolean network (BN) as a computational model for gene regulatory networks (GRN) (Kauffman, 1969). BN has been applied in many fields, including chemistry, biology, social networks, economics and computer science. Examples are given in Akutsu, Hayashida, Ching, and Ng (2007), Albert and Barabási (2000), Aldana (2003), Heidel, Maloney, Farrow, and Rogers (2003), Shmulevich, Dougherty, and Zhang (2002) and references therein. In recent years, a new mathematical tool called the semi-tensor product (STP) of matrices (Cheng, Qi, & Li, 2011a) has resolved many challenging BN problems, such as stability and stabilization (Cheng, Qi, Li, & Liu, 2011; Li & Wang, 2013; Li, Wang, & Liu, 2014; Qi, Cheng, & Hu, 2010), controllability (Cheng & Qi, 2009; Li & Sun, 2011; Liu, Chen, Lu, & Wu, 2015; Lu, Zhong, Ho, Tang, & Cao, 2016), synchronization of BNs (Li & Chu, 2012), disturbance decoupling (Cheng, 2011), identification (Cheng & Zhao, 2011), pinning control (Lu, Zhong, Huang, & Cao, 2016) and optimal control (Zhao, Li, & Cheng, 2011). STP has also been applied to logical dynamic systems by Cheng (2009), Cheng, Feng, and Lv (2012), Cheng and Qi (2010), Cheng, Qi, and Li (2011b), Cheng and Zhao (2011), Li and Cheng (2010), Li, Sun, and Wu (2011), Wu, Kumar, and Shen (2016), Xu and Hong Zhao et al. (2011) and Zhao, Qi, and Cheng (2010).

Observability is an important concept in control engineering and systems biology (Cobelli & Romanin-Jacur, 1976). Observability determines whether the initial state can be identified from output measurements. In a linear time-invariant (LTI) system, the observability matrix relates the vector formed by stacking the output measurements over a period of time to the initial state. Consequently, an LTI system is observable if and only if the observability matrix is of full rank. However, this criterion is inapplicable to BNs under the same definition of the observability matrix. Cheng and Qi (2009) proved that a BN is observable if and only if no two columns of the observability matrix are identical. However, Laschov, Margaliot, and Even (2013) showed that the observability matrix of an observable BN can be singular.

For a BN in algebraic form, the state vector of the entire network is the STP of the states of all nodes. Therefore, to establish the relationship between the output measurements and the initial state in algebraic form, we can multiply all output measurements over a period of time using the STP, rather than simply stacking them up. This leads to a new definition of the observability matrix for BNs that is constructed using the Khatri–Rao (K–R) product of matrices (Ljung & Söderström, 1983), termed as the K–R observability matrix in this paper. The K–R observability matrix of a BN is a logical matrix, and can be calculated by a simple algorithm. In this paper, we show that a BN is observable if and only if the K–R observability matrix is of full rank; equivalently, if the number of nonzero rows of the K–R observability matrix equals the dimension of the algebraic-form state vector.

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Table 1
Notations.

Notations	Definitions
Δ_n	Set of columns of I_n
$[n:m]$	Set of integers x with $n \le x \le m$
$\mathscr{B}_{n\times m}$	Set of $n \times m$ Boolean matrices
$\mathscr{L}_{n\times m}$	Set of $n \times m$ logical matrices
$\mathscr{S}^T(x)$	set $\{y \in \Delta_n \mid y^T x \neq 0\}$
$\text{Col}_i(A)$	ith column of matrix A
$\text{Row}_j(A)$	jth row of matrix A
$\text{NRow}_j(A)$	jth nonzero row of matrix A
$\mathbf{r}(A)$	Number of nonzero rows of matrix A
δ_n^i	ith column of I_n
$\delta_n[i_1 i_2 \cdots i_m]$	Matrix A with $\text{Col}_s(A) = \delta_n^{i_s}$
$\ltimes$	Semi-tensor product of matrices
$*$	Khatri–Rao product of matrices, $\text{Col}_j(A * B) := \text{Col}_j(A) \ltimes \text{Col}_j(B)$
$\lvert\mathcal{M}\rvert$	Cardinal number of set $\mathcal{M}$

The exact initial state of an unobservable BN cannot be identified by output measurements. However, the state space can be partitioned into a collection of subsets. The subset containing the initial state can then be identified from the output measurements. In other words, we can partition the state space such that any two distinct initial states selected, respectively, from two different subsets are distinguishable. In this paper, such a partition is called a distinguishable partition of the state space. For a given partition, the distinguishability can be checked by the K–R observability matrix. In practice, we are more interested in the finest distinguishable partition (FDP) of the state space, i.e., the distinguishable partition that is finer than any other distinguishable partition. The FDP characterizes the finest identification of the initial states from the output measurements. We show that for the FDP, any distinct initial states selected from a same subset must be indistinguishable. We then prove that the number of subsets contained in the FDP equals the rank of the K–R observability matrix, and that each nonzero row of the K–R observability matrix represents one subset.

Observability index is an important concept for general LTI systems, as it characterizes the least number of successive output measurements needed for the initial state identification. As far as our knowledge, there is no similar definition for BNs in the existing literature. In this paper, in terms of the K–R observability matrix, we define the observability index of a BN and show that it characterizes the least number of successive output measurements required for the finest identification of initial states.

This work is closely related to Cheng, Qi, Liu, and Wang (2016), Cheng and Zhao (2011), Fornasini and Valcher (2013), Zhao et al. (2010) and Zhang, Zhang, and Xie (2016), where the observability of Boolean control networks (BCNs) were addressed from different perspectives.

The remainder of this paper is arranged as follows. Section 2 defines the K–R observability matrix, proposes the observability index, and develops the algorithm for calculating the K–R observability matrix. Section 3 investigates the distinguishability of state space partitions. In Section 4, we propose an algorithm for the FDP and find a necessary and sufficient condition for BN observability. The observability index is also explained in this section. Section 5 illustrates the results in this paper with examples. Concluding remarks are presented in Section 6. The notations used in this paper are listed in Table 1.

2. Observability matrix and observability index

2.1. Basic results

A Boolean network (BN) in algebraic form is described as follows (Cheng et al., 2011a):

$$\begin{cases} x(t+1) = Lx(t) \\ y(t) = Hx(t) \end{cases} \tag{1}$$

where $x = x_1 \ltimes x_2 \ltimes \cdots \ltimes x_n \in \Delta_{2^n}$ and $y = y_1 \ltimes y_2 \ltimes \cdots \ltimes y_q \in \Delta_{2^q}$ denote the state and output variables of the network, respectively. Here, $x_i \in \Delta_2$ and $y_j \in \Delta_2$ are the states of the ith state node and the jth output node, respectively. $L \in \mathscr{L}_{2^n\times 2^n}$ and $H \in \mathscr{L}_{2^q\times 2^n}$ are the structural and output matrices, respectively. The solution to BN (1) with initial state $x_0 \in \Delta_{2^n}$ is denoted by $x(t; x_0)$. The output is denoted by $y(t; x_0)$; that is, $y(t; x_0) = Hx(t; x_0)$. For convenience, we define $Y(t; x_0) := [y(0; x_0)^T \ y(1; x_0)^T \ \cdots \ y(t; x_0)^T]^T$ and denote by $\mathbf{y}(t; x_0)$ the algebraic form of $Y(t; x_0)$, i.e., $\mathbf{y}(t; x_0) := y(0; x_0) \ltimes y(1; x_0) \ltimes \cdots \ltimes y(t; x_0)$.

Definition 1 (Cheng & Qi, 2009)**.** Two distinct initial states x_0 and $\bar{x}_0$ are said to be distinguishable on $[0:N]$ if $\mathbf{y}(N, x_0) \neq \mathbf{y}(N; \bar{x}_0)$; equivalently, if $Y(N, x_0) \neq Y(N; \bar{x}_0)$. BN (1) is said to be observable on $[0:N]$ if any two distinct initial states are distinguishable on $[0:N]$.

Definition 2 (Cheng & Qi, 2009; Laschov et al., 2013)**.** The observability matrix of BN (1) on $[0:\nu-1]$ is defined as $\mathcal{O}_\nu := [H^T \ (HL)^T \ \cdots \ (HL^{\nu-1})^T]^T$.

In terms of the above definition, we have $Y(N; x_0) = \mathcal{O}_{N+1}x_0$.

Theorem 1 (Cheng & Qi, 2009)**.** *BN (1) is observable on* $[0:N]$ *if and only if no two columns of* $\mathcal{O}_{N+1}$ *are identical.* □

In the following example, we show that an observable BN can have a singular observability matrix.

Example 1 (Laschov et al., 2013)**.** Consider a BN with $n = 2, q = 1$, $L = \delta_4[4\ 2\ 1\ 3]$ and $H = \delta_2[1\ 1\ 2\ 2]$. A simple calculation shows that the observability matrix $\mathcal{O}_2 = \begin{bmatrix} H \\ HL \end{bmatrix} = \begin{bmatrix} \delta_2[1\ 1\ 2\ 2] \\ \delta_2[2\ 1\ 1\ 2] \end{bmatrix}$ is singular, but the BN is observable on $[0:1]$ by Theorem 1.

2.2. K–R observability matrix and observability index

Definition 3. The K–R observability matrix on $[0:\nu-1]$ is defined as $\mathcal{O}_\nu^* := H * (HL) * \cdots * (HL^{\nu-1})$.

This new definition satisfies $\mathbf{y}(N; x_0) = \mathcal{O}_{N+1}^* x_0$. By (1), we actually have $\mathbf{y}(N; x_0) = y(0; x_0) \ltimes y(1; x_0) \ltimes \cdots \ltimes y(N; x_0) = (Hx_0) \ltimes (HLx_0) \ltimes \cdots \ltimes (HL^N x_0) = \left[H * (HL) * \cdots * (HL^N)\right] x_0 = \mathcal{O}_{N+1}^* x_0$.

In the following, we consider a logical matrix L with $\mathbf{r}(L)$ nonzero rows. The following result is obviously true.

Proposition 1. *For any logical matrix* L*, it holds that* $\mathbf{r}(L) = \text{rank}(L)$. □

Definition 4. The observability index of BN (1) is defined as $\nu_o := \min\{N \mid \mathbf{r}(\mathcal{O}_N^*) = \mathbf{r}(\mathcal{O}_{N+j}^*)\ \forall j \ge 0\}$.

2.3. Calculation of K–R observability matrix

Proposition 2. *Suppose that* $L = \delta_{2^n}[i_1, i_2, \ldots, i_{2^n}]$, $H = \delta_{2^q}[j_1, j_2, \ldots, j_{2^n}]$, *and suppose that a K–R observability matrix takes the form* $\mathcal{O}_\nu^* = \delta_{2^{q\nu}}[l_{\nu,1}, l_{\nu,2}, \ldots, l_{\nu,2^n}]$. *Then* $l_{\nu,s}$, $s = 1, 2, \ldots, 2^n$, *can be calculated by the following iteration:*

$$\begin{cases} j_{0,s} = j_s \\ l_{1,s} = j_s \\ j_{\nu,s} = j_{\nu-1,i_s} \\ l_{\nu+1,s} = 2^q(l_{\nu,s} - 1) + j_{\nu,s} \end{cases} \tag{2}$$