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Brief paper

A family of piecewise affine control Lyapunov functions ☆

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ABSTRACT

This paper presents a novel method to construct a family of piecewise affine control Lyapunov functions. Unlike most of existing methods which require the contractivity of their domain of definition, the proposed control Lyapunov functions are defined over a so-called N-step controllable set, which is known not to be contractive. Accordingly, a robust control design procedure is presented which only requires solving a linear programming problem at each sampling time. The construction is finally illustrated via a numerical example.

1. Introduction

As a fundamental concept in control theory (Lyapunov, 1907), Lyapunov stability has been applied in intensive studies related to stability analysis as well as control design. For the design purpose, control Lyapunov functions are usually employed to synthesize controllers guaranteeing closed-loop stability in the sense of Lyapunov, see among the others Khalil (2002) and Zubov and Boron (1964). Such control Lyapunov functions are usually chosen a priori with special structural properties. More clearly, in the case of linear optimal control, suitable quadratic objective functions represent control Lyapunov candidates, see e.g. Anderson and Moore (2007), Chmielewski and Manousiouthakis (1996) and Daafouz and Bernussou (2001). Moreover, model predictive control (MPC) usually employs finite/infinite horizon quadratic cost functions as control Lyapunov candidates, see for instance Kothare, Balakrishnan, and Morari (1996) and Mayne, Rawlings, Rao, and Scokaert (2000). Extensive studies about control Lyapunov functions for nonlinear systems have been found in the literature, see among the others Primbs, Nevistić, and Doyle (1999). In case the underlying system is subject to constraints, such a control Lyapunov function should be determined such that the recursive feasibility is ensured. This problem is closely related to the determination of the domain of attraction.

Piecewise linear control Lyapunov functions date back to the studies in Gutman and Cwikel (1987) for the nominal case, and are subsequently extended for the robust case to cope with additive disturbances and/or polytopic uncertainty in Blanchini (1994), Rakovic and Baric (2010) and Nguyen, Gutman, Olaru, and Hovd (2013), leading to simple design formulations as linear programming problems. However, these studies require that such control Lyapunov functions be defined over contractive sets to guarantee its strict decrease and the recursive feasibility.

This paper aims to present the construction of a more general family of control Lyapunov candidates in the context of constrained control, namely piecewise affine functions. These candidates are defined over a so-called N-step controllable set for a given positive integer N, which is obtained from an increasing sequence of N polytopes, and is known to be not necessarily (one step) contractive. Accordingly, we prove that the conditions of a Lyapunov function (the positivity and the strict decrease) are satisfied within this N-step controllable set with a suitable robust control algorithm. Note that the complexity of the proposed control Lyapunov functions increases as N becomes larger, leading to a more complex control algorithm over the ones using contractive set in Blanchini (1994) and Nguyen, Olaru, Rodriguez-Ayerbe, and Kvasnica (2017), since the number of constraints in the proposed algorithm is larger than the ones in two latter references. However, since this method only requires solving a linear program at each sampling instant, these results can be used for constrained control systems with fast dynamics, e.g. vibration attenuation system, cf. Gulan, Takács, Nguyen, Olaru, Rodriguez-Ayerbe, and Rohal'-Ilkiv (2017a, 2017b).

2. Generalities and basic notions

Throughout the paper, $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{N}$, $\mathbb{N}_{>0}$ denote the field of real numbers, the set of nonnegative real numbers, the set of nonnegative integers and the positive integer set, respectively. The following index set is also defined, for ease of presentation, with

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respect to a given $N \in \mathbb{N}_{>0}$: $\mathcal{I}_N = \{1, 2, \ldots, N\}$. A polyhedron is defined as the intersection of finitely many closed halfspaces. A polytope is defined as a bounded polyhedron. Also, $\mathcal{V}(P)$ is understood as the set of vertices of polytope P. The distance from a point $x \in \mathbb{R}^d$ to a set $S \subset \mathbb{R}^d$ denoted by $\rho_S(x)$ is defined as $\rho_S(x) := \min_{y \in S} \sqrt{(x-y)^T(x-y)}$. Given two sets $S_1, S_2 \subset \mathbb{R}^d$, we define the set $S_1 \setminus S_2 := \left\{x \in \mathbb{R}^d : x \in S_1,\ x \notin S_2\right\}$. Also, the Minkowski sum of these two sets is defined as $S_1 \oplus S_2 := \{x_1 + x_2 : x_1 \in S_1, x_2 \in S_2\}$. We use ∂S to denote the boundary of a compact set S. Also, $\mathrm{int}(S)$ represents the interior of a full-dimensional set S and $\mathrm{conv}(S)$ denotes its convex hull. A function $\alpha(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be of class $\mathcal{K}$, if it is continuous, strictly increasing and $\alpha(0) = 0$.

3. Problem settings

In this paper, we consider a linear time-varying system:

$$x_{k+1} = A(k)x_k + B(k)u_k + w_k, \tag{1}$$

where x_k, u_k, w_k denote the state, control variable and additive disturbance at time k. These variables satisfy

$$x_k \in \mathbb{X},\ \ u_k \in \mathbb{U},\ \ w_k \in \mathbb{W}, \tag{2}$$

where the constraint sets $\mathbb{X} \subset \mathbb{R}^{d_x}, \mathbb{U} \subset \mathbb{R}^{d_u}, \mathbb{W} \subset \mathbb{R}^{d_x}$ are assumed to be polytopes, containing the origin in their interior, with given $d_x, d_u \in \mathbb{N}_{>0}$. This assumption ensures that the origin as the equilibrium point satisfies the above constraints. In case the origin is not the equilibrium point, the system can be translated into the frame of the equilibrium point and the construction can be easily adapted. This assumption will be used later in (4) to build a control Lyapunov function which is only equal to 0 at the equilibrium point. Also, the state–space matrices $A(k), B(k)$ are assumed to belong to a given polytope, defined as below:

$$[A(k)\, B(k)] \in \Psi := \mathrm{conv}\left\{[A_1\ B_1], \ldots, [A_L\ B_L]\right\}. \tag{3}$$

This paper aims to construct a new family of control Lyapunov functions, also referred to as convex liftings in the present framework. In particular, the control Lyapunov functions presented in this paper are more general than the piecewise linear family proposed in Blanchini (1994) and Rakovic and Baric (2010), since besides their convex, piecewise affine properties, they are defined over the *N-step controllable set*, known to be non-necessarily contractive.

4. Construction of control Lyapunov functions

Before describing the main result, we need to recall some important ingredients which are instrumental for the proposed construction of control Lyapunov functions. Positive invariance concept has been investigated in many studies (Aubin & Cellina, 2012; Bitsoris, 1988a, 1988b; Bitsoris & Vassilaki, 1995; Blanchini & Miani, 2007) and used in different control strategy designs. In case the system is affected by disturbances, the robust positive invariance concept is of use instead.

Definition 4.1. Given an admissible linear control law $u_k = Kx_k \in \mathbb{U}$, a set $\Omega \subseteq \mathbb{X}$ is called *robust positively invariant* with respect to system (1) subject to (2) and (3) iff

$$(A(k) + B(k)K)\Omega \oplus \mathbb{W} \subseteq \Omega,\ \ \forall\, [A(k)\ B(k)] \in \Psi.$$

In order to compute such a set Ω, one should determine a local controller $u = Kx \in \mathbb{U}$, which can cope with both the model uncertainty (3) and additive disturbances in $\mathbb{W}$. Such a gain K can be obtained by different methods, see among the others Daafouz and Bernussou (2001) and Kothare et al. (1996). According to such a local controller, one can use existing algorithms to compute a robust positively invariant set, e.g., in Gilbert and Tan (1991), Kolmanovsky and Gilbert (1998) and Nguyen et al. (2013). The definition of another important ingredient *N-step controllable set* is recalled below.

Definition 4.2. Consider system (1) subject to model uncertainty (3) and constraints (2). Given a robust positively invariant set Ω and $N \in \mathbb{N}_{>0}$, a set denoted by $\mathcal{K}_N(\Omega) \subseteq \mathbb{X}$ is called the *N-step controllable set* if any point belonging to this set can reach Ω in N steps in the presence of suitable admissible controller, while staying inside $\mathbb{X}$ despite any disturbances in $\mathbb{W}$ and model uncertainties in Ψ. It is mathematically defined below for all $i \in \mathcal{I}_N$:

$$\mathcal{K}_0(\Omega) = \Omega,$$
$$\mathcal{K}_i(\Omega) = \Big\{x_k \in \mathbb{X} :\ \exists u_k \in \mathbb{U}\ \text{ s.t. }\ \forall\, [A(k)\ B(k)] \in \Psi,$$
$$(A(k)x_k + B(k)u_k) \oplus \mathbb{W} \subseteq \mathcal{K}_{i-1}(\Omega)\Big\}.$$

To determine $\mathcal{K}_N(\Omega)$, the computation of the 1-step controllable set should be performed, i.e. $\mathcal{K}_i(\Omega)$ should be computed according to $\mathcal{K}_{i-1}(\Omega)$. Indeed, if one defines an intermediate set

$$S := \Big\{\left[x^T\ u^T\right]^T \in \mathbb{R}^{d_x+d_u} : x \in \mathbb{X},\ \ u \in \mathbb{U},$$
$$(A_j x + B_j u) \oplus \mathbb{W} \subseteq \mathcal{K}_{i-1}(\Omega),\ \ \forall j \in \mathcal{I}_L\Big\},$$

then $\mathcal{K}_i(\Omega)$ can be determined as the orthogonal projection of the set S defined above onto the space of x. Similar computation is repeated until $i = N$ to obtain $\mathcal{K}_N(\Omega)$. The interested reader is referred to Section 2.6 in Kerrigan (2001) for further detail. One can easily see that if Ω is empty, then so are $\mathcal{K}_i(\Omega)$ for $i \in \mathbb{N}$. Consequently, the proposed method cannot apply in this case. Also, if $\Omega \neq \emptyset$ is not of full-dimension, then $\mathcal{K}_i(\Omega)$ might not be of full-dimension either. To illustrate this point, we consider the following simple system: $x_{k+1} = \begin{bmatrix} 1 & 0 \\ 0 & \alpha_k \end{bmatrix} x_k + \begin{bmatrix} 1 \\ \alpha_k \end{bmatrix} u_k$, where uncertainty $\alpha_k \in [-1,\ 1]$ and control variable $u_k \in [-1,\ 1]$. If $\Omega = \{0\}$, then one can easily compute $\mathcal{K}_i(\Omega) = \left\{[y_1\ y_2]^T \in \mathbb{R}^2 :\ y_1 = y_2 \in [-1,\ 1]\right\}$ for all $i \in \mathbb{N}_{>0}$. Although the proposed method can still apply in this case, we exclusively consider the case as presented in Assumption 1 to ensure that $\mathcal{K}_i(\Omega)$ for $i \in \mathbb{N}$ are of full-dimension.

Assumption 1. Ω is a full-dimensional polytope in $\mathbb{R}^{d_x}$.

Note that $0 \in \mathrm{int}(\Omega)$ since the origin is assumed to be the equilibrium point and a full-dimensional set Ω is robust positively invariant. Furthermore, since Ω satisfies Assumption 1 and $\mathbb{X}, \mathbb{U}, \mathbb{W}$ are polytopes, then $\mathcal{K}_i(\Omega)$ for any finite $i \in \mathbb{N}$ is also a full-dimensional polytope. Therefore, the existence of a full-dimensional $\mathcal{K}_N(\Omega)$ depends on the existence of a full-dimensional Ω, since they fulfill the following property.

Lemma 4.1. *Given a robust positively invariant set Ω satisfying Assumption 1, then $\mathcal{K}_{i-1}(\Omega) \subseteq \mathcal{K}_i(\Omega)$ for all $i \in \mathbb{N}$.*

Clearly, the sequence $\{\mathcal{K}_i(\Omega)\}_{i=0}^{\infty}$ is increasing and bounded above by $\mathbb{X}$, accordingly the limit exists. Note that if the limit of this sequence is finitely determined, there exists $N^* \in \mathbb{N}_{>0}$ such that $\mathcal{K}_{N^*-1}(\Omega) \subset \mathcal{K}_{N^*}(\Omega) = \mathcal{K}_{N^*+1}(\Omega)$. In this case, any positive integer $N < N^*$ is suitable for the proposed construction to avoid $\mathcal{V}(\mathcal{K}_{N+1}(\Omega)) \setminus \mathcal{K}_N(\Omega) = \emptyset$. Otherwise, if the limit of $\{\mathcal{K}_i(\Omega)\}_{i=0}^{\infty}$ is not finitely determined, this end may not be a polytope. In this case, one can always ensure for any $N < +\infty$ that $\mathcal{K}_N(\Omega)$ is a polytope and $\mathcal{V}(\mathcal{K}_{N+1}(\Omega)) \setminus \mathcal{K}_N(\Omega) \neq \emptyset$. As a consequence, any positive integer N can be used in the proposed construction.

Before presenting the main results of the paper, a parametric linear programming (pLP) problem is recalled in the sequel:

$$\max_x\ c^T x\ \ \text{s.t.}\ \ Hx \le G\lambda + b,$$