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Brief paper

Power spectrum identification for quantum linear systems ☆

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ABSTRACT

We investigate system identification for general quantum linear systems in the situation where the input field is prepared as stationary (squeezed) quantum noise. In this regime the output field is characterised by the power spectrum, which encodes covariance of the output state. We address which parameters can be identified from the power spectrum and how to construct a system realisation from the power spectrum. The power spectrum depends on the system parameters via the transfer function. We show that the transfer function can be uniquely recovered from the power spectrum, so that equivalent systems are related by a symplectic transformation.

1. Introduction

System identification theory (Burgarth & Yuasa, 2012; Gambetta & Wiseman, 2001; Gammelmark, Julsgaard, & Mølmer, 2013; Guţă & Kiukas, 2015, 2016; Ljung, 1987; Mabuchi, 1996; Pintelon & Schoukens, 2012) lies at the interface between control theory and statistical inference, and deals with the estimation of unknown parameters of dynamical systems and processes from input–output data.

In this paper we consider system identification for *quantum linear systems* (QLSs). QLSs are a class of models used in quantum optics, opto-mechanical systems, electro-dynamical systems, cavity QED systems and elsewhere (Doherty & Jacobs, 1999; Gardiner & Zoller, 2004; Koga & Yamamoto, 2012; Stockton, van Handel, & Mabuchi, 2004; Tian, 2012; Walls & Milburn, 2007). They have many applications, such as quantum memories, entanglement generation, quantum information processing and quantum control (Bouten, Van Handel, & James, 2007; Dong & Petersen, 2010; James, Nurdin, & Petersen, 2008; Nurdin & Gough, 2014; Wiseman & Milburn, 2009; Yamamoto, 2014).

QLSs are examples of input–output models. Typically, one has access to the field and is able to prepare an input. After the coupling, the parameters of the system (black-box) are imprinted on the output. In a nutshell, the system identification problem is to estimate dynamical parameters from the output data, obtained by performing measurements on the output. The identification of QLSs is by now a well developed subject in 'classical' systems theory (Anderson, Newcomb, Kalman, & Youla, 1966; Glover & Willems, 1974; Ho & Kalman, 1966; Kalman, 1963; Ljung, 1987; Pintelon & Schoukens, 2012; Youla, 1961; Zhou, Doyle, Glover, et al., 1996), but has not been fully explored in the quantum domain (Guţă & Yamamoto, 2016). We distinguish two contrasting approaches to the identification of QLSs .

In the first approach, one probes the system with a known *time-dependent* input signal (e.g. coherent state), then uses the output measurement data to compute an estimator of the dynamical parameter(s). In this setting the *transfer function* entirely encapsulates the systems input–output behaviour. Therefore, the basic identifiability problem is to find the class of systems with the same transfer function. This problem has been addressed, firstly for the special class of *passive* QLSs in Guţă and Yamamoto (2016) and then for general QLSs in Levitt and Guţă (2017). In particular, it was seen that *minimal* systems with the same transfer function are related by symplectic transformations on the space of system modes.

The second approach and the one we consider here is to probe the systems with *time-stationary* pure Gaussian states with independent increments, i.e. squeezed vacuum noise. This setup is relevant when it may not be possible for the experimenter to use time-dependent inputs, e.g. in modelling a system where one can only observe the power spectrum of the output and assumes that the spectrum has been generated by some time-stationary pure Gaussian state input. If the system is minimal and Hurwitz stable, the dynamics exhibits an initial transience period after which it

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reaches stationarity and the output is in a stationary Gaussian state, whose covariance in the frequency domain is given by the *power spectrum*. The power spectrum depends quadratically on the transfer function, so the parameters which are identifiable in the stationary scenario will also be identifiable in the time-dependent one. Our goal is to understand to what extent the converse is also true. This problem is of the type: 'for a square rational matrix $V(s)$, where $s \in \mathbb{C}$ find rational matrix $W(s)$ such that $V(s) = W(s)W(-\bar{s})^\dagger$ for all $s \in \mathbb{C}$, which in the classical literature is called the *spectral factorisation problem* (Anderson et al., 1966). Levitt and Guţă (2017) looked at this problem for a generic class of single input single output (SISO) QLSs. Now, for a given minimal system there may exist lower dimensional systems with the same power spectrum. To understand this, consider the system's stationary state and note that it can be uniquely written as a tensor product between a pure and a mixed Gaussian state (cf. the symplectic decomposition Wolf (2008)). Restricting the system to the mixed component leaves the power spectrum unchanged (Levitt & Guţă, 2017). Conversely, if the stationary state is fully mixed, there exists no smaller dimensional system with the same power spectrum. We call such systems *globally minimal*.

The main result here is to show that under global minimality the power spectrum determines the transfer function, and therefore the equivalence classes are the same as those in the transfer function. It is interesting to note that this equivalence is a consequence of the unitarity and purity of the input state, and does not hold for a generic classical linear system (Anderson et al., 1966; Glover & Willems, 1974). The key to our proof is in reducing the power spectrum identifiability problem to an equivalent transfer function identifiability problem.

This paper is organised as follows: In Section 2 we review the setup of input–output QLS. In Section 3 we outline the power spectrum identifiability problem. We introduce the notion of global minimality for systems and review recent important results. Our main identifiability result is presented in Section 4, cf., Theorem 4. Finally, we outline a method to construct a globally minimal system realisation from the power spectrum.

We use the following notations: For a matrix $X = (X_{ij})$, let $\overline{X} = (X_{ij}^*)$, $X^T = (X_{ji})$, $X^\dagger = (X_{ji}^*)$ represent the complex conjugation, transpose and adjoint matrix respectively, where '*' indicates complex conjugation. We also use the 'doubled-up notation' $\breve{X} := \left[X^T, \overline{X}^T\right]^T$ and $\Delta(A, B) := \left[\begin{smallmatrix} A & B \\ \bar{B} & \bar{A}\end{smallmatrix}\right]$. For a matrix $Z \in \mathbb{R}^{2n\times 2m}$ define $Z^\flat = J_m Z^\dagger J_n$, where $J_n = \left[\begin{smallmatrix} \mathbb{1}_n & 0 \\ 0 & -\mathbb{1}_n\end{smallmatrix}\right]$. A similar notation is used for matrices of operators. We use '$\mathbb{1}$' to represent the identity matrix or operator. δ_{jk} is Kronecker delta and $\delta(t)$ is Dirac delta. The commutator is denoted by $[\cdot, \cdot]$.

Definition 1. A matrix $S \in \mathbb{C}^{2m\times 2m}$ is said to be $\flat$-*unitary* if it is invertible and satisfies $S^\flat S = SS^\flat = \mathbb{1}_{2m}$. If additionally, S is of the form $S = \Delta(S_-, S_+)$ for some $S_-, S_+ \in \mathbb{C}^{m\times m}$ then we say that it is *symplectic*.

2. Quantum linear systems

In this section we briefly review the QLS theory. We refer to Gardiner and Zoller (2004) for a more detailed discussion on the input–output formalism, and to the reviews (Nurdin, James, & Doherty, 2009; Petersen, 2016) for the theory of QLSs.

2.1. Time-domain representation

A quantum input–output system is defined as a continuous variables system coupled to a Bosonic environment, such that their joint evolution is linear in all canonical variables. The system is

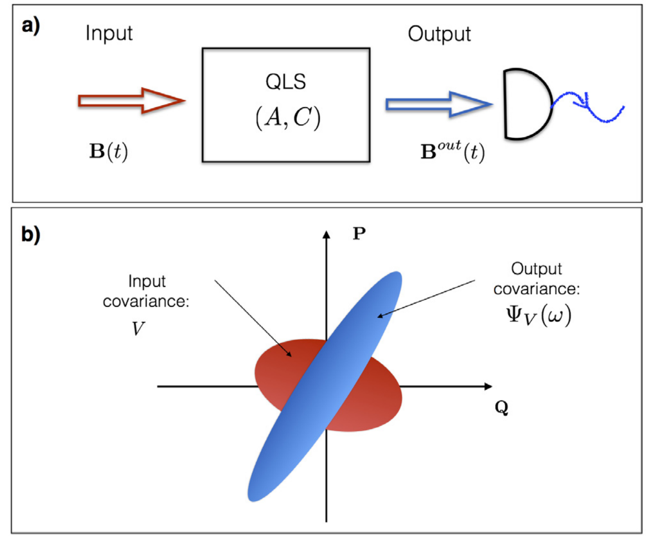

Fig. 1. (a) System identification problem: find parameters (A, C) of a linear input–output system by measuring output. (b) Stationary scenario: *power spectrum* describes output covariance in frequency domain.

described by the column vector of annihilation operators, $\mathbf{a} := [\mathbf{a}_1, \ldots, \mathbf{a}_n]^T$, representing the n cv modes (see Fig. 1a). Together with their respective creation operators $\mathbf{a}^* := [\mathbf{a}_1^*, \ldots, \mathbf{a}_n^*]^T$ they satisfy the canonical commutation relations (CCR) $\left[\mathbf{a}_i, \mathbf{a}_j^*\right] = \delta_{ij}\mathbb{1}$. We denote by $\mathcal{H} := L^2(\mathbb{R}^n)$ the Hilbert space of the system carrying the standard representation of the n modes. The environment is modelled by m bosonic fields, called *input channels*, whose fundamental variables are the fields $\mathbf{B}(t) := [\mathbf{B}_1(t), \ldots, \mathbf{B}_m(t)]^T$, where $t \in \mathbb{R}$ represents time. The fields satisfy the CCR $\left[\mathbf{b}_i(t), \mathbf{b}_j^*(s)\right] = \delta(t-s)\delta_{ij}\mathbb{1}$, where $\mathbf{b}_i(t)$ are the infinitesimal (white noise) annihilation operators formally defined as $\mathbf{b}_i(t) := d\mathbf{B}_i(t)/dt$ (Petersen, 2016). The operators can be defined in a standard fashion on the Fock space $\mathcal{F} = \mathcal{F}(L^2(\mathbb{R}) \otimes \mathbb{C}^m)$ (Bouten et al., 2007). We consider the scenario where the input is prepared in a *pure, stationary in time, mean-zero, Gaussian state* with independent increments with covariance matrix

$$\left\langle \begin{pmatrix} d\mathbf{B}(t)d\mathbf{B}(t)^\dagger & d\mathbf{B}(t)d\mathbf{B}(t)^T \\ d\mathbf{B}^*(t)d\mathbf{B}(t)^\dagger & d\mathbf{B}^*(t)d\mathbf{B}(t)^T \end{pmatrix}\right\rangle = \begin{pmatrix} N^T+\mathbb{1} & M \\ M^\dagger & N \end{pmatrix} dt$$
$$:= V(N, M)dt, \tag{1}$$

where the brackets denote a quantum expectation. Note that $N = N^\dagger$, $M = M^T$ and $V \geq 0$, which ensures that the state does not violate the uncertainty principle. In particular, $N = M = 0$ corresponds to the vacuum state, while pure squeezed states satisfy $\overline{M}(N+1)^{-1}M = N$ (Gough, James, & Nurdin, 2010).

The dynamics of a general input–output system is determined by the system's Hamiltonian and coupling to the environment. In the Markov approximation, the joint unitary evolution of system and environment is given by the (interaction picture) unitary $\mathbf{U}(t)$ on the joint space $\mathcal{H} \otimes \mathcal{F}$, which is the solution of the quantum stochastic differential equation (Bouten et al., 2007; Gardiner & Zoller, 2004; Hudson & Parthasarathy, 1984)

$$d\mathbf{U}(t) = \Big(-(\mathbf{K} + i\mathbf{H})\,dt + \mathbf{L}d\mathbf{B}^\dagger(t) - \mathbf{L}^\dagger d\mathbf{B}(t)\Big)\,\mathbf{U}(t), \tag{2}$$

where $\mathbf{K} = \frac{1}{2}\left(\mathbf{L}^\dagger(1+N^T)\mathbf{L} + \mathbf{L}^T N\bar{\mathbf{L}} - \mathbf{L}^\dagger M\bar{\mathbf{L}} - \mathbf{L}^T\overline{M}\mathbf{L}\right)$ and initial condition $\mathbf{U}(0) = \mathbf{I}$. Here, $\mathbf{H}$ and $\mathbf{L}$ are system operators describing the system's Hamiltonian and field coupling.