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Brief paper

Cascade and locally dissipative realizations of linear quantum systems for pure Gaussian state covariance assignment☆

Shan Ma [a,b,*], Matthew J. Woolley [a], Ian R. Petersen [c], Naoki Yamamoto [d]

[a] School of Engineering and Information Technology, University of New South Wales at the Australian Defence Force Academy, Canberra ACT 2600, Australia
[b] School of Information Science and Engineering, Central South University, Changsha 410083, China
[c] Research School of Engineering, The Australian National University, Canberra ACT 2601, Australia
[d] Department of Applied Physics and Physico-Informatics, Keio University, Yokohama 223-8522, Japan

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ABSTRACT

This paper presents two realizations of linear quantum systems for covariance assignment corresponding to pure Gaussian states. The first one is called a cascade realization; given any covariance matrix corresponding to a pure Gaussian state, we can construct a cascaded quantum system generating that state. The second one is called a locally dissipative realization; given a covariance matrix corresponding to a pure Gaussian state, if it satisfies certain conditions, we can construct a linear quantum system that has only local interactions with its environment and achieves the assigned covariance matrix. Both realizations are illustrated by examples from quantum optics.

1. Introduction

For stochastic systems, many of the performance objectives are expressed in terms of the variances (or covariances) of the system variables. One way to achieve these objectives is to assign an appropriate matrix value to the covariance of the vector of system variables. This method, referred to as *covariance assignment*, has been extensively studied in a series of papers, e.g., in Hotz and Skelton (1987) and Skelton and Ikeda (1989). For linear stochastic systems with white noises, the covariance matrix can be computed by solving a Lyapunov equation for the system. In this case, the covariance assignment problem reduces to designing system matrices such that the corresponding Lyapunov equation has a prescribed solution.

Turning our attention to the quantum case, we find that the covariance matrix plays an essential role as well in the field of quantum information. In particular for a *linear quantum system*, the importance of the covariance matrix stands out, because it can fully characterize the *entanglement* property, which is crucial for conducting quantum information processing (Braunstein & Pati, 2003; Weedbrook et al., 2012). Therefore it should be of great use to investigate the covariance assignment problem for linear quantum systems. There are several such proposals; Ohki, Hara, and Yamamoto (2011) study a quantum feedback control problem for covariance assignment, and Ikeda and Yamamoto (2013), Koga and Yamamoto (2012), Ma, Petersen, and Woolley (2017), Ma, Woolley, Petersen, and Yamamoto (2014, 2017) and Yamamoto (2012) analyze systems that generate a *pure* Gaussian state. Since a Gaussian state (with zero mean) is uniquely determined by its covariance matrix, the aforementioned covariance assignment problem is also known as the Gaussian state generation problem; thus, if a linear quantum system achieves a covariance matrix corresponding to a target Gaussian state, we call that the system generates this Gaussian state.

Let us especially focus on Refs. Ikeda and Yamamoto (2013), Koga and Yamamoto (2012), and Yamamoto (2012), which provide the basis of this paper. As mentioned before, in those papers pure Gaussian states are examined, which are a particularly important subclass of Gaussian states such that the highest performance of Gaussian quantum information processing can be realized (Braunstein & Pati, 2003; Menicucci, Flammia, & van Loock, 2011; Menicucci et al., 2006; Weedbrook et al., 2012). Then they provided several methods to construct a stable linear

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* Corresponding author.
E-mail addresses: shanma.adfa@gmail.com (S. Ma), m.woolley@adfa.edu.au (M.J. Woolley), i.r.petersen@gmail.com (I.R. Petersen), yamamoto@appi.keio.ac.jp (N. Yamamoto).

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quantum system generating a given pure Gaussian state. Moreover, necessary and sufficient conditions for generating an arbitrary pure entangled Gaussian state are given in Koga and Yamamoto (2012) and Yamamoto (2012); these are important results, because such a state serves as an essential resource for Gaussian quantum information processing tasks. In the literature several methods for generating various pure entangled Gaussian states have been proposed. For instance, Adesso (2006) gives a systematic method to generate an arbitrary pure entangled Gaussian state; the idea is to construct a *coherent* process by applying a sequence of prescribed unitary operations (composed of beam splitters (BS) and squeezers in the optics case) to an initial state. Thus, this method is essentially a *closed*-system approach. In contrast, the approach we take here is an *open*-system one; that is, we aim to construct coherent and dissipative processes such that the system is stable and evolves into a desired target pure Gaussian state. This strategy is categorized as a so-called *reservoir engineering* method (Cirac, Parkins, Blatt, & Zoller, 1993; Krauter et al., 2011; Ockeloen-Korppi et al., 2017; Poyatos, Cirac, & Zoller, 1996; Wang & Clerk, 2013; Woolley & Clerk, 2014); in general, this approach has a clear advantage that the system has good robustness properties with respect to initial states and evolution time.

Now we describe the problem considered in this paper. The methods developed in Koga and Yamamoto (2012) and Yamamoto (2012) lead to infinitely many linear quantum systems that generate a target pure Gaussian state. Some of these systems are easy to implement, while others are not. Then a natural question is how to find a linear quantum system that is simple to implement, while still generates the desired pure Gaussian state.

In this paper, we provide two convenient realizations of a linear quantum system generating a target pure Gaussian state. The first one is a *cascade realization*, which is a typical system structure found in the literature (Gardiner, 1993; Nurdin, 2010; Petersen, 2011). We show that given any covariance matrix corresponding to a pure Gaussian state, we can construct a cascaded quantum system generating that state. This cascaded system is a series connection of several subsystems in which the output of one is fed as the input to the next. A clear advantage of the cascade realization is that these subsystems can be placed at remote sites. Note that the cascade structure has also been widely studied in the classical control literature, e.g., Huang, James, and Jiang (2005).

The second one is a *locally dissipative realization*, which is motivated by the specific system structure found in, e.g. Ikeda and Yamamoto (2013), Kraus et al. (2008), Rafiee, Lupo, Mokhtari, and Mancini (2012) and Ticozzi and Viola (2012). In these references the notion of *quasi-locality* has been studied, but in this paper we focus on a stronger notion, *locality*. Here "locally dissipative" means that all the system–environment interactions act only on one system component. Implementations of locally dissipative systems should be considerably easier than those systems which have non-local interactions (Bachor & Ralph, 2004). In this paper, we show that given a covariance matrix corresponding to a pure Gaussian state, if it satisfies certain conditions, we can construct a locally dissipative quantum system generating that state.

Lastly, we mention that the state generated by our method is an *internal* one confined in the system (e.g., an intra-cavity state in quantum optics), rather than an external optical field state. It means that if we aim to perform some quantum information processing with that Gaussian state, it must be extracted to the outside by for instance the method developed in Tufarelli, Ferraro, Serafini, Bose, and Kim (2014). In particular, by applying some non-Gaussian operations (e.g., the cubic phase gate or photon counting) on that extracted Gaussian state, we can realize quantum information processing tasks such as entanglement distillation and universal quantum computation (Weedbrook et al., 2012). On the other hand, a generated internal Gaussian state is not necessarily extracted to the outside for the purpose of precision measurement in the scenario of quantum metrology; for example, in magnetometry a spin squeezed state of an atomic ensemble can be directly used and in this case the generated internal Gaussian state is not necessarily extracted to the outside (Tóth & Apellaniz, 2014).

Notation. For $A = [A_{jk}]$, we define $A^\top = [A_{kj}]$, $A^\dagger = [A_{kj}^*]$, where the superscript * denotes either the complex conjugate of a complex number or the adjoint of an operator. $\mathrm{diag}[\tau_1, \ldots, \tau_n]$ denotes an $n \times n$ diagonal matrix with τ_j, $j = 1, 2, \ldots, n$, on its main diagonal. $\mathscr{P}_N$ is a $2N \times 2N$ permutation matrix defined by $\mathscr{P}_N[x_1\ x_2\ x_3\ x_4\ \cdots\ x_{2N}]^\top = [x_1\ x_3\ \cdots\ x_{2N-1}\ x_2\ x_4\ \cdots\ x_{2N}]^\top$ for any column vector $[x_1\ x_2\ x_3\ x_4\ \cdots\ x_{2N}]^\top$.

2. Preliminaries

We consider a linear quantum system G of N modes. Each mode is characterized by a pair of quadrature operators $\{\hat{q}_j, \hat{p}_j\}$, $j = 1, 2, \ldots, N$. Collecting them into an operator-valued vector $\hat{x} \triangleq \begin{bmatrix}\hat{q}_1 & \cdots & \hat{q}_N & \hat{p}_1 & \cdots & \hat{p}_N\end{bmatrix}^\top$, we write the canonical commutation relations as

$$\left[\hat{x}, \hat{x}^\top\right] \triangleq \hat{x}\hat{x}^\top - \left(\hat{x}\hat{x}^\top\right)^\top = i\Sigma, \quad \Sigma \triangleq \begin{bmatrix} 0 & I_N \\ -I_N & 0 \end{bmatrix}. \tag{1}$$

Here we emphasize that the transpose operation $\top$, when applied to an operator-valued matrix (say, $\hat{x}\hat{x}^\top$), only exchanges the indices of the matrix and leaves the entries unchanged. Therefore $\left(\hat{x}\hat{x}^\top\right)^\top \neq \hat{x}\hat{x}^\top$. Let $\hat{H}$ be the Hamiltonian, and let $\{\hat{c}_j\}$, $j = 1, 2, \ldots, K$, be Lindblad operators that represent the interactions between the system and its environment. For convenience, we collect all the Lindblad operators as an operator-valued vector $\hat{L} = \begin{bmatrix}\hat{c}_1 & \hat{c}_2 & \cdots & \hat{c}_K\end{bmatrix}^\top$ and call $\hat{L}$ the *coupling vector*. Suppose $\hat{H}$ is quadratic in $\hat{x}$, i.e., $\hat{H} = \frac{1}{2}\hat{x}^\top M\hat{x}$, with $M = M^\top \in \mathbb{R}^{2N\times 2N}$, and $\hat{L}$ is linear in $\hat{x}$, i.e., $\hat{L} = C\hat{x}$, with $C \in \mathbb{C}^{K\times 2N}$, then the quantum system G can be described by the following quantum stochastic differential equations (QSDEs)

$$\begin{cases} d\hat{x}(t) = \mathscr{A}\hat{x}(t)dt + \mathscr{B}\left[d\hat{A}^\top(t)\ d\hat{A}^\dagger(t)\right]^\top, \\ d\hat{Y}(t) = \mathscr{C}\hat{x}(t)dt + d\hat{A}(t), \end{cases} \tag{2}$$

where $\mathscr{A} = \Sigma(M + \mathrm{Im}(C^\dagger C))$, $\mathscr{B} = i\Sigma[-C^\dagger\ \ C^\top]$, $\mathscr{C} = C$ (Yamamoto, 2012), (Wiseman & Milburn, 2010, Chapter 6). The input $d\hat{A}(t) = \begin{bmatrix} d\hat{A}_1(t) & \cdots & d\hat{A}_K(t)\end{bmatrix}^\top$ represents K independent quantum stochastic processes, with $d\hat{A}_j(t), j = 1, 2, \ldots, K$, satisfying the following quantum Itō rules:

$$\begin{cases} d\hat{A}_j(t)d\hat{A}_k^*(t) = \delta_{jk}dt, \\ d\hat{A}_j(t)d\hat{A}_k(t) = d\hat{A}_j^*(t)d\hat{A}_k^*(t) = d\hat{A}_j^*(t)d\hat{A}_k(t) = 0, \end{cases} \tag{3}$$

where δ_{jk} is the Kronecker δ-function. The output $d\hat{Y}(t) = \begin{bmatrix} d\hat{Y}_1(t) & \cdots & d\hat{Y}_K(t)\end{bmatrix}^\top$ satisfies quantum Itō rules similar to (3) (Belavkin, 1992; Bouten, Handel, & James, 2007; Gardiner & Zoller, 2000; Hudson & Parthasarathy, 1984; Wiseman & Milburn, 2010; Yamamoto, 2012). The quantum expectation of the vector $\hat{x}$ is denoted by $\langle\hat{x}\rangle$ and the covariance matrix is given by $V = \frac{1}{2}\langle\triangle\hat{x}\triangle\hat{x}^\top + (\triangle\hat{x}\triangle\hat{x}^\top)^\top\rangle$, where $\triangle\hat{x} = \hat{x} - \langle\hat{x}\rangle$; see, e.g., Koga and Yamamoto (2012), Menicucci et al., (2011) and Yamamoto (2012). The time evolutions of the mean vector $\langle\hat{x}(t)\rangle$ and the covariance matrix $V(t)$ can be derived from (2) by using the quantum Itō rule. They are given by

$$\begin{cases} \dfrac{d\langle\hat{x}(t)\rangle}{dt} = \mathscr{A}\langle\hat{x}(t)\rangle, & \text{(a)} \\ \dfrac{dV(t)}{dt} = \mathscr{A}V(t) + V(t)\mathscr{A}^\top + \dfrac{1}{2}\mathscr{B}\mathscr{B}^\dagger. & \text{(b)} \end{cases} \tag{4}$$