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Brief paper

$\mathscr{H}_-$ index for discrete-time stochastic systems with Markovian jump and multiplicative noise[☆]

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ABSTRACT

In this paper, we discuss the $\mathscr{H}_-$ index problem for stochastic linear discrete-time systems subject to Markovian jump and multiplicative noise, for which, a necessary and sufficient condition for an $\mathscr{H}_-$ index larger than $\gamma > 0$ is given in finite time horizon. It is shown that the $\mathscr{H}_-$ index larger than a given value is equivalent to the solvability of a certain generalized difference Riccati equation (GDRE). What we have obtained generalizes the results of deterministic systems to stochastic models. Moreover, the $\mathscr{H}_-$ index problem for *square* systems in infinite horizon is also studied. Finally, some examples are presented to illustrate the effectiveness of the proposed theoretical results.

1. Introduction

It is well-known that almost every control process in practice, is inevitably influenced by unknown inputs (disturbances and noises) as well as various faults arising from sensors, actuators or process components. The sensitivity to fault is normally measured by $\mathscr{H}_-$ index in the fault detection field. $\mathscr{H}_-$ index is the measurement of the minimum influence of the fault on the residual signal, which can be regarded as the worst case fault sensitivity and has been widely applied to $\mathscr{H}_-/\mathscr{H}_\infty$ fault detection filter design (Frank & Ding, 1997).

Fault detection for complex dynamic systems has been becoming more and more important due to the considerations of securities, reliabilities and fault tolerances. During the last three decades, a great deal of attention has been paid to model-based fault detection (Chen & Pattor, 1999; Frank & Ding, 1997; Pattor, 1997). The main purpose of fault detection is to construct a filter called "residual generator" which is sensitive to faults but is robust to unknown inputs (Chen & Pattor, 1999). Up to now, some effective criteria such as $\mathscr{H}_\infty$ norm, $\mathscr{H}_2$ norm and $\mathscr{H}_-$ index have been presented in Chen and Pattor (1999), Ding, Jeinsch, Frank, and Ding (2000), Hou and Pattor (1996), Iwasaki, Hara, and Yamauchi (2003), Jaimoukha, Li, and Papakos (2006), Liu, Wang, and Yang (2005) and Pattor (1997). All these criteria lead to multiple objective optimization problems such as $\mathscr{H}_2/\mathscr{H}_\infty$, $\mathscr{H}_\infty/\mathscr{H}_\infty$ and $\mathscr{H}_-/\mathscr{H}_\infty$ that can be found in Khan, Abid, and Ding (2014), Li and Liu (2013a, b), Li, Mo, and Zhou (2010), Li and Zhou (2009), Liu and Zhou (2007, 2008) and Zhong, Ding, and Ding (2010). Among these indices, the fault detection filter based on $\mathscr{H}_-/\mathscr{H}_\infty$ is of more advantageous to diagnosing the potential fault in early time, where the maximum influence of the unknown input on the residual signal is measured by $\mathscr{H}_\infty$ norm, while the $\mathscr{H}_-$ index measures the minimum influence of the fault on the residual signal; see Jaimoukha et al. (2006), Hou and Pattor (1996), Iwasaki et al. (2003), Khan et al. (2014), Li and Liu (2013a, b), Li et al. (2010), Li and Zhou (2009), Liu et al. (2005), Liu and Zhou (2007, 2008), Zhong et al. (2010), and the references therein.

In Chen and Pattor (1999), the definition of $\mathscr{H}_-$ index was given via the minimum non-zero singular value at zero frequency. Based on the generalized KYP lemma, the $\mathscr{H}_-$ index in finite frequency was discussed in Iwasaki et al. (2003). Liu et al. (2005) extended the definition of $\mathscr{H}_-$ index to the minimum singular value over all frequency range. In recent years, much attention has been devoted to $\mathscr{H}_-$ index in the time domain. In Li and Zhou (2009), a fault detection generator in finite time horizon was designed to maximize the fault sensitivity by means of $\mathscr{H}_-$ index. The authors of Li et al. (2010) and Zhong et al. (2010) researched the optimal fault detection for linear discrete time-varying systems, and other characterizations of $\mathscr{H}_-$ index can be found in Li and Liu (2013a, b) for

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linear continuous time-varying systems and discrete time-varying systems. The work in Khan et al. (2014) was to design a fault detection filter for nonlinear discrete-time systems using $\mathscr{H}_-/\mathscr{H}_\infty$ approach. The fault detection filter design for linear time-invariant systems subject to disturbance and faults was considered in Li, Liu, and Jiang (2015b), while Li, Liu, and Jiang (2015a) designed the fault detection filter based on optimization and partial decoupling.

Although there is much work on the $\mathscr{H}_-$ index of deterministic systems (Khan et al., 2014; Li & Liu, 2013a, b; Li et al., 2015a, b; Liu & Zhou, 2007, 2008), few results have been reported for stochastic systems. It is well-known that, in practical modeling, stochastic disturbances often occur, hence, stochastic control theory has attracted many researchers' attention for a long time (Chen & Zheng, 2013; Lin & Zhang, 2015; Wei, Wu, & Karimi, 2016; Wu, 2015; Zhang & Chen, 2012; Zhang, Lin, & Chen, 2017; Zhang, Zhao, & Sheng, 2015; Zhao & Deng, 2014, 2015). In order that the designed fault detection filter can work reliably, we need to consider stochastic disturbances in constructing mathematical models. Due to great application of Markovian switching systems, the related research for such systems has become an active topic; see, e.g., Chen and Zheng (2014), Costa and de Oliveira (2012), Dragan, Morozan, and Stoica (2010), Lin, Zhang, Niu, and Liu (2011), Qin, Liang, Yang, Pan, and Yang (2016), Yan, Zhang, and Zhang (2015) and Zhang, Shi, and Lin (2016). However, little attention has been paid to $\mathscr{H}_-$ index for stochastic Markov jump systems with multiplicative noise, which motivates us to do this study.

The contribution and novelty of this paper is as follows: Compared with the existing results, we extend the definition and the characteristic of $\mathscr{H}_-$ index from determinate systems to stochastic cases. The appearances of Markov jumps and multiplicative noises lead to a high difficulty in mathematical derivations. Our first main result provides a necessary and sufficient condition for the $\mathscr{H}_-$ index for stochastic linear discrete time-varying systems with Markovian jump and multiplicative noise in finite time horizon. It is shown that the $\mathscr{H}_-$ index greater than a given value is equivalent to the solvability of a set of generalized difference Riccati equations(GDREs). By the solvability of a set of generalized algebraic Riccati equations (GAREs), the second main result gives a necessary and sufficient condition of the $\mathscr{H}_-$ index for stochastic linear discrete-time *square* systems in infinite time horizon. Our obtained results have a potential application to the fault detection filter design. Combining stochastic $\mathscr{H}_\infty$ theory with our given results, we may study the mixed stochastic $\mathscr{H}_-/\mathscr{H}_\infty$ fault detection filter, which is our future work.

The outline of this paper is arranged as below: Section 2 is devoted to developing some efficient criteria for the $\mathscr{H}_-$ index of stochastic linear discrete time-varying systems in finite horizon. In Section 3, the $\mathscr{H}_-$ index of stochastic linear discrete-time *square* systems in infinite horizon is discussed. Section 4 contains some examples provided to show the efficiency of the presented results. Finally, Section 5 ends this paper with a brief conclusion.

Notation: R^m is the m-dimensional Euclidean space with 2-norm $\|\cdot\|$. $R^{m\times n}$ is the vector space of all $m\times n$ matrices with entries in R. $\mathcal{S}_n(R)$ is the set of all real symmetric matrices $R^{n\times n}$. A' denotes the transpose of the complex matrix A. A^{-1} is the inverse of A. We denote a positive semi-definite (positive definite) matrix A by $A\geq 0$ ($A>0$). E denotes the mathematical expectation operator. χ_A is the indicator function of a set A. I_n is $n\times n$ identity matrix. $0_{n\times m}$ is $n\times m$ zero matrix. $N_T=\{0,1,\ldots,T\}$, $N=\{0,1,\ldots\}$. $\bar{L}=\{1,2,\ldots,l\}$. A *tall* (respectively, *wide*, *square*) system refers to its input dimensions less than (respectively, more than, equal to) its output dimensions.

2. Finite horizon stochastic $\mathscr{H}_-$ index

In this section, we will present a necessary and sufficient condition for the $\mathscr{H}_-$ index which is crucial to the analysis of the fault detection filter. Consider the following stochastic system $\mathcal{G}$:

$$\begin{cases} x(t+1)=A_{r(t)}(t)x(t)+B_{r(t)}(t)v(t) \\ \qquad\qquad +[\tilde{A}_{r(t)}(t)x(t)+\tilde{B}_{r(t)}(t)v(t)]w(t), \\ z(t)=C_{r(t)}(t)x(t)+D_{r(t)}(t)v(t),\ t\in N_T \end{cases} \tag{1}$$

where $\{w(t)\}_{t\in N_T}$ is a sequence of one-dimensional white noises defined on the complete probability space $(\Omega,\mathcal{F},\mathcal{P})$, with $E[w(t)]=0$ and $E[w(s)w(t)]=\delta_{st}$, $s,t\in N_T$, where δ_{st} is the Kronecker delta. $x(t)\in R^n$, $v(t)\in R^m$ and $z(t)\in R^q$ are system state, control input and controlled output, respectively. $r(t)$ is a non-homogeneous Markov chain taking value in $\bar{L}$ with the transition probability matrix $\mathcal{P}(t)=(p_{ij}(t))_{l\times l}$, $p_{ij}(t)=\mathcal{P}(r(t+1)=j|r(t)=i)$. $p_i(t)=\mathcal{P}\{r(t)=i\}$, $i\in\bar{L}$. $\{w(t)\}_{t\in N_T}$ is independent of $\{r(t)\}_{t\in N_T}$ for $t\in N_T$. We define the σ-algebra $\mathcal{F}_t$ generated by $w(\cdot)$ and $r(\cdot)$ as $\mathcal{F}_t=\sigma\{w(s),r(j):s\in N_{t-1},j\in N_t\}$, $t\in N_T$. $\{\mathcal{F}_t\}_{t\in N_T}$ is an increasing sequence of σ-algebras with $\mathcal{F}_t\subset\mathcal{F}$. Let $L^2(\Omega,R^m)$ be the space of R^m-valued random vectors ξ with $E\|\xi\|^2<\infty$, and $l^2_w(N_T;R^m)$ denote the space of all sequences $\{z(t)\}_{t\in N_T}$ such that $z(t)\in L^2(\Omega,R^m)$ is $\mathcal{F}_t$-measurable with the l^2-norm satisfying $\|z(\cdot)\|_{l^2_w(N_T;R^m)}=(\sum_{t=0}^T E\|z(t)\|^2)^{1/2}<\infty$. For simplicity, we suppose that x_0 is deterministic. For any $T\in N$ and $(x_0,v)\in R^n\times l^2_w(N_T;R^m)$, there exists a unique solution $x(\cdot)=x(\cdot;x_0,r_0,v)\in l^2_w(N_{T+1};R^m)$ of (1) with $x(0)=x_0$, $r(0)=r_0$.

For convenience, we define the following operators. Let $P_i(t)\in\mathcal{S}_n(R)$, $t\in N_{T+1}$, $i\in\bar{L}$, define

$$\Phi_i^t=\sum_{j=1}^l p_{ij}(t)P_j(t+1),\ \Phi_i(P)=\sum_{j=1}^l p_{ij}(t)P_j,$$

$$L_i(t,P)=A_i(t)'\Phi_i^tA_i(t)+\tilde{A}_i(t)'\Phi_i^t\tilde{A}_i(t)+C_i(t)'C_i(t),$$

$$K_i(t,P)=A_i(t)'\Phi_i^tB_i(t)+\tilde{A}_i(t)'\Phi_i^t\tilde{B}_i(t)+C_i(t)'D_i(t),$$

$$R_i(t,P)=B_i(t)'\Phi_i^tB_i(t)+\tilde{B}_i(t)'\Phi_i^t\tilde{B}_i(t),$$

$$H_i^\gamma(t,P)=R_i(t,P)+D_i(t)'D_i(t)-\gamma^2I_m,$$

$$L_i(P(t))=A_i'\Phi_i^tA_i+\tilde{A}_i'\Phi_i^t\tilde{A}_i+C_i'C_i,$$

$$K_i(P(t))=A_i'\Phi_i^tB_i+\tilde{A}_i'\Phi_i^t\tilde{B}_i+C_i'D_i,$$

$$R_i(P(t))=B_i'\Phi_i^tB_i+\tilde{B}_i'\Phi_i^t\tilde{B}_i,$$

$$H_i^\gamma(P(t))=R_i(P(t))+D_i'D_i-\gamma^2I_m,$$
$$L_i(P)=A_i'\Phi_i(P)A_i+\tilde{A}_i'\Phi_i(P)\tilde{A}_i+C_i'C_i,$$
$$K_i(P)=A_i'\Phi_i(P)B_i+\tilde{A}_i'\Phi_i(P)\tilde{B}_i+C_i'D_i,$$
$$R_i(P)=B_i'\Phi_i(P)B_i+\tilde{B}_i'\Phi_i(P)\tilde{B}_i,$$
$$H_i^\gamma(P)=R_i(P)+D_i'D_i-\gamma^2I_m.$$

Definition 1. For system (1), given $0<T<\infty$, define

$$\|\mathcal{G}\|_-^{[0,T]}=\inf_{v\in l^2_w(N_T;R^m),v\neq 0,r_0\in\bar{L},x_0=0}\frac{\|z(t)\|_{l^2_w(N_T;R^q)}}{\|v(t)\|_{l^2_w(N_T;R^m)}}, \tag{2}$$

which is called the $\mathscr{H}_-$ index of (1) in N_T.

Remark 1. From Definition 1, the smallest sensitivity of system (1) from the input v to the output z can be described by the $\mathscr{H}_-$ index