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Realization of time-delay systems ☆

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ABSTRACT

The paper addresses the problem of transforming a single-input single-output nonlinear retarded time-delay system, described by an input–output equation, in the traditional observable state space form. The solution is generalized from the delay-free case and depends on integrability of certain submodule of differential 1-forms. The integrability conditions are improved to make them constructive. Finally, it is explained why one may obtain two realizations, which are not connected by bi-causal change of state coordinates.

1. Introduction

Numerous papers address the realization problem for nonlinear delay-free control systems, see Belikov, Kotta, and Tõnso (2014, 2015), Zhang, Moog, and Xia (2010) and the references therein. The same cannot be said about nonlinear time-delay systems, where up to the authors knowledge only the paper (Garcia-Ramirez, Moog, Califano, & Márquez-Martínez, 2016) addresses the special case of linear realization up to nonlinear input–output injection term. The problem has been also studied for linear time-delay systems (Glusing-Luersen, 1997) and for the case when the delay depends on the state (Verriest, 2013). However, the reverse problem, i.e., obtaining the i/o equations via state elimination has been already addressed in Anguelova and Wennberg (2009) and Halas and Anguelova (2013). It has been shown in Kotta, Kotta, onso, and Halas (2011) that a nonlinear single-input single-output (SISO) delay-free input–output (i/o) equation is realizable in the state–space form if and only if certain vector space of differential 1-forms is integrable. In the present paper this result is generalized for nonlinear retarded SISO time-delay systems with commensurable delays, i.e., for delays that are multiples of some fixed minimal delay. Extension is, however, not direct since time-delay systems are infinite dimensional. It means that the differential 1-forms have to be viewed as elements of a module, and not a vector space. A consequence is that full rank conditions are not equivalent to invertibility (matrices over ring may have full rank, but nevertheless be non-invertible within the same set of matrices). Therefore, different system properties may be generalized in (often two) different ways. One example is observability property, see Anguelova and Wennberg (2010), Garcia-Ramirez et al. (2016), Xia, Márquez-Martínez, Zagalak, and Moog (2002), Zheng and Richard (2016) and the references therein. In Garcia-Ramirez et al. (2016) one distinguishes weak and strong observability, where weak observability corresponds to full rank of certain matrix, and strong observability to invertibility of the matrix. The same situation happens when one speaks about integrability of the modules of 1-forms. Frobenius theorem is no longer appropriate, since it provides only rather restrictive sufficient conditions. In Kaldmäe, Califano, and Moog (2016) integrability problem is studied for nonlinear time-delay systems. Two notions – weak and strong integrability – are defined and characterized. However, no constructive method is given to check the necessary and sufficient condition of strong integrability. In this paper we improve the results of Kaldmäe et al. (2016) and present a directly verifiable necessary and sufficient condition. This condition is needed to solve the realization problem in the time-delay case.

The aim of this paper is to transform a SISO retarded time-delay system, described by the i/o equation, into a state–space form, which is strongly or weakly observable. As shown already in Garcia-Ramirez et al. (2016), two realizations are not necessarily connected by bi-causal change of coordinates. The reasons are explained in this paper; the source of the problem is in two observability notions. The problem can be avoided when we require the state equations to be strongly observable.

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The paper is organized as follows. In Section 2 the sequence of submodules is defined in terms of which the realization problem will be studied. Integrability of 1-forms is studied in Section 3 and the main results on realization problem are presented in Section 4. The paper ends with conclusions.

2. Preliminaries

In this paper we work with nonlinear retarded single-input single-output (SISO) systems with constant commensurable delays described by the input–output equation of the form

$$\Phi(y^{(n)}(t), y^{(n-1)}(t-i), \ldots, y(t-i), u^{(n-1)}(t-i), \ldots, u(t-i); 0 \le i \le s) = 0, \tag{1}$$

where Φ is analytic. Denote by $\mathcal{A}$ the ring of analytic functions depending on finite number of variables from the set $\mathcal{C} = \{y^{(k)}(t-i), u^{(k)}(t-i); i, k \in \mathbb{N}\}$. The delay operator δ is defined on $\mathcal{A}$ as $\delta\varphi(\xi(t)) = \varphi(\xi(t-1))$, where $\varphi \in \mathcal{A}$ and $\xi(t) \in \mathcal{C}$. Let I be the minimal ideal of $\mathcal{A}$ that contains Φ, all the derivatives of Φ and $\delta^i\Phi$ for all $i > 0$. Now, one can construct the quotient ring $\mathcal{A}/I$, where the addition and multiplication are defined in a natural way. We assume that I is prime, which means that $\mathcal{A}/I$ is an integral domain and thus allows to construct the field of fractions of the ring $\mathcal{A}/I$, denoted by $\mathcal{K}$. The operator δ is extended to $\mathcal{K}$ in a natural way.

Define the vector space of 1-forms as $\mathcal{E} = \text{span}_{\mathcal{K}}\{\text{d}\varphi | \varphi \in \mathcal{K}\}$. The operator δ is extended to $\mathcal{E}$ as $\delta(\sum_j a_j \text{d}\xi_j) = \sum_j \delta(a_j)\text{d}\delta(\xi_j)$, where $a_j \in \mathcal{K}$ and $\xi_j \in \mathcal{C}$. Using the delay operator δ, a non-commutative polynomial ring $\mathcal{K}[\vartheta]$ can be constructed. The addition is defined in $\mathcal{K}[\vartheta]$ as usual, but for multiplication the following rule is used: $\vartheta\varphi = \delta(\varphi)\vartheta$ for $\varphi \in \mathcal{K}$. Now, the 1-forms may be alternatively viewed as elements of the module $\mathcal{M} = \text{span}_{\mathcal{K}[\vartheta]}\{\text{d}\varphi | \varphi \in \mathcal{K}\}$. Unlike a vector space, not every module has a basis. The modules, that do have a basis, are called free modules. Since $\mathcal{K}[\vartheta]$ satisfies the left Ore condition (Xia et al., 2002), any two basis of a free module have the same cardinality, which is called the rank of the free module.

Definition 1 (*Xia et al., 2002*)**.** The closure of a free submodule $\mathcal{F}$ of $\mathcal{M}$, denoted by $cl_{\mathcal{K}[\vartheta]}(\mathcal{F})$, is defined as $cl_{\mathcal{K}[\vartheta]}(\mathcal{F}) = \{\omega \in \mathcal{M} \mid \exists p(\vartheta) \in \mathcal{K}[\vartheta], \text{ s.t. } p(\vartheta)\omega \in \mathcal{F}\}$.

By definition, the closure of the free submodule $\mathcal{F}$ is the largest free submodule, containing $\mathcal{F}$, and having the same rank as $\mathcal{F}$. If the closure of the submodule $\mathcal{F}$ is equal to itself, then $\mathcal{F}$ is said to be closed.

We also use the set of matrices $\mathcal{K}[\vartheta]^{r\times l}$ defined over the polynomial ring $\mathcal{K}[\vartheta]$. A special subset of $\mathcal{K}[\vartheta]^{r\times r}$ is the set of unimodular matrices, denoted by $\mathcal{U}_r[\vartheta]$. A matrix $U \in \mathcal{K}[\vartheta]^{r\times r}$ is said to be unimodular if it has an inverse in $\mathcal{K}[\vartheta]^{r\times r}$. A useful property for polynomial matrices in $\mathcal{K}[\vartheta]^{r\times l}$ is the Jacobson decomposition, see Cohn (1965).

Theorem 1 (*Cohn, 1965*)**.** *For every $M(\vartheta) \in \mathcal{K}[\vartheta]^{r\times l}$, $r \le l$, there exist matrices $V(\vartheta) \in \mathcal{U}_r[\vartheta]$ and $U(\vartheta) \in \mathcal{U}_l[\vartheta]$ such that*

$$V(\vartheta)M(\vartheta)U(\vartheta) = (\Delta_r, 0_{r,l-r}), \tag{2}$$

where $0_{r,l-r}$ is the matrix with zero entries, Δ_r is square diagonal matrix with elements $(\sigma_1, \ldots, \sigma_k, 0, \ldots, 0)$ such that $\sigma_i \in \mathcal{K}[\vartheta]$, for $i = 1, \ldots, k$, and σ_i is a divisor of σ_{i+1} for all $i = 1, \ldots, k-1$, i.e., $\sigma_{i+1} = \alpha\sigma_i$ for some $\alpha \in \mathcal{K}[\vartheta]$.

Note that the matrices $U(\vartheta)$ and $V(\vartheta)$ in (2) are not unique whereas Δ_r is. The matrix $(\Delta_r, 0_{r,l-r})$ is called the Jacobson form of the matrix $M(\vartheta)$.

To make the presentation more compact, the following notations are introduced: $\xi_{[s]} = (\xi(t), \ldots, \xi(t-s))$ for all $\xi \in \mathcal{C}$. Thus, the system (1) can be rewritten as

$$\Phi(y^{(n)}, y^{(n-1)}_{[s]}, \ldots, y_{[s]}, u^{(n-1)}_{[s]}, \ldots, u_{[s]}) = 0. \tag{3}$$

Also, for time-derivatives and time-delays the following notations are used: $\text{d}/\text{d}t\xi = \dot{\xi}$, $\text{d}^2/\text{d}t^2\xi = \ddot{\xi}$, $\xi(t-i) = \xi^{[-i]}$ for $i > 0$.

In Xia et al. (2002) a sequence $\{\mathcal{H}_i; i \ge 1\}$ of submodules of $\mathcal{M}$ is used to study the accessibility property of time delay systems. Here we define similar sequence for systems of the form (3) as

$$\begin{aligned} \mathcal{H}_1 &= \text{span}_{\mathcal{K}[\vartheta]}\{\text{d}y^{(n-1)}, \ldots, \text{d}y, \text{d}u^{(n-1)}, \ldots, \text{d}u\} \\ \mathcal{H}_{i+1} &= \{\omega \in \mathcal{H}_i | \omega^{(1)} \in \mathcal{H}_i\}. \end{aligned} \tag{4}$$

It has been shown in Xia et al. (2002) that sequence (4) converges to a submodule, denoted by $\mathcal{H}_\infty$, and all the submodules $\mathcal{H}_i$ are closed. From now on we assume that $\mathcal{H}_\infty = \{0\}$, which guarantees that the system (3) is accessible (Xia et al., 2002). Note that, by definition, $\mathcal{H}_2 = \text{span}_{\mathcal{K}[\vartheta]}\{\text{d}y^{(n-1)}, \ldots, \text{d}y, \text{d}u^{(n-2)}, \ldots, \text{d}u\}$. Now, if we know two consecutive submodules $\mathcal{H}_{i-1}$ and $\mathcal{H}_i$, then Algorithm 1 can be used to compute $\mathcal{H}_{i+1}$.

Algorithm 1. Denote by ρ_i the rank of submodule $\mathcal{H}_i$ and let $\mathcal{H}_{i-1} = \text{span}_{\mathcal{K}[\vartheta]}\{\eta_1, \ldots, \eta_{\rho_i}, \mu_1, \ldots, \mu_{\rho_{i-1}-\rho_i}\}$, $\mathcal{H}_i = \text{span}_{\mathcal{K}[\vartheta]}\{\eta_1, \ldots, \eta_{\rho_i}\}$.

1. Compute $\dot{\eta}_j$ for $j = 1, \ldots, \rho_i$. By the definition of $\mathcal{H}_i$, $\dot{\eta}_j = \sum_{l=1}^{\rho_i} a_{j,l}\eta_l + \sum_{\sigma=1}^{\rho_{i-1}-\rho_i} c_{j,\sigma}\mu_\sigma$ for some $a_{j,l}, c_{j,\sigma} \in \mathcal{K}[\vartheta]$.
2. Construct a matrix $C \in \mathcal{K}[\vartheta]^{\rho_i\times(\rho_{i-1}-\rho_i)}$ whose elements are $c_{j,\sigma}$, $j = 1, \ldots, \rho_i$, $\sigma = 1, \ldots, \rho_{i-1} - \rho_i$.
3. Find the left-kernel $B \in \mathcal{K}[\vartheta]^{(\rho_i-\gamma)\times\rho_i}$ of the matrix C, where γ is the rank of matrix C.
4. Define the basis elements of $\mathcal{H}_{i+1}$ as $B\eta$, where $\eta = (\eta_1, \ldots, \eta_{\rho_i})^T$.

The subspaces $\mathcal{H}_i$ have the following properties.

Lemma 1. *(i) The submodule $\mathcal{H}_i$ of system (3) has rank $2n+1-i$.*
(ii) $\omega \in \mathcal{E}$ is an element of $\mathcal{H}_i$ iff $\omega^{(i-1)} \in \mathcal{H}_1$.

Proof. *(i)* The proof is by mathematical induction. Since in Algorithm 1 $\rho_{i-1} = 2n-1+2$ and $\rho_i = 2n-i+1$, the matrix C has dimension $(2n-i+1)\times 1$ and the matrix B dimension $(2n-i)\times(2n-i+1)$.

(ii) By the definition of the sequence $\{\mathcal{H}_i; i \ge 1\}$ $\omega \in \mathcal{H}_i \Leftrightarrow \dot{\omega} \in \mathcal{H}_{i-1} \Leftrightarrow \cdots \Leftrightarrow \omega^{(i-1)} \in \mathcal{H}_1$. ■

3. Integrability

Compared to the results in Kaldmäe et al. (2016), we give a different necessary and sufficient condition for checking the strong integrability of a set of 1-forms. The new condition gives a method to check the condition in Kaldmäe et al. (2016) and is thus more constructive.

Definition 2 (*Kaldmäe et al., 2016*)**.** A set of 1-forms $\{\omega_1, \ldots, \omega_k\}$, independent over $\mathcal{K}[\vartheta]$, is said to be strongly (weakly) integrable if there exist k independent functions $\{\varphi_1, \ldots, \varphi_k\}$, such that $\text{span}_{\mathcal{K}[\vartheta]}\{\omega_1, \ldots, \omega_k\} = \text{span}_{\mathcal{K}[\vartheta]}\{\text{d}\varphi_1, \ldots, \text{d}\varphi_k\}$ ($\text{span}_{\mathcal{K}[\vartheta]}\{\omega_1, \ldots, \omega_k\} \subseteq \text{span}_{\mathcal{K}[\vartheta]}\{\text{d}\varphi_1, \ldots, \text{d}\varphi_k\}$).

If the set of 1-forms $\{\omega_1, \ldots, \omega_k\}$ is strongly (respectively weakly) integrable, then the submodule $\text{span}_{\mathcal{K}[\vartheta]}\{\omega_1, \ldots, \omega_k\}$ is said to be strongly (respectively weakly) integrable.

Next, the conditions for checking strong integrability of a set of 1-forms are developed. Define for $p \ge 0$ the sequence of vector