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Dominant eigenvalue minimization with trace preserving diagonal perturbation: Subset design problem☆

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ABSTRACT

Motivated by network resource allocation needs, we study the problem of minimizing the dominant eigenvalue of an essentially-nonnegative matrix with respect to a trace-preserving or fixed-trace diagonal perturbation, in the case where only a subset of the diagonal entries can be perturbed. Graph-theoretic characterizations of the optimal subset design are obtained: in particular, the design is connected to the structure of a reduced effective graph defined from the essentially-nonnegative matrix. Also, the change in the optimum is studied when additional diagonal entries are constrained to be undesignable, from both an algebraic and graph-theoretic perspective. These results are developed in part using properties of the Perron complement of nonnegative matrices, and the concept of line-sum symmetry. Some results apply to general essentially-nonnegative matrices, while others are specialized for sub-classes (e.g., diagonally-symmetrizable, or having a single node cut).

1. Introduction

The problem of allocating or redistributing limited local control resources to shape a network's dynamics is of interest in several domains, including in the mitigation of network spread processes, management of various compartmental systems, and control of transients in large-scale infrastructures. In many of these application domains, control resources can only be placed or recruited in a limited subset of network locations. The limited control resources thus must be designed to leverage the intrinsic interconnectivity of the network, so as to meet performance criteria. Further, the scale and complexity of the networks often dictate that simple topological rubrics rather than formal methods are needed for resource allocation. Also, in many of these application domains, resource redesign as constraints change is often needed in lieu of or in addition to *ab initio* design.

The purpose of this paper is to study a canonical optimization problem which arises in the design of limited control resources to shape an associated network dynamics. Specifically, a network dynamics defined by an essentially-nonnegative (Metzler) matrix – a matrix whose off-diagonal entries are nonnegative – is considered. Placement of local control resources is abstracted as perturbing diagonal entries of the Metzler matrix (altering local dynamical characteristics). The goal of the design is to optimize this diagonal perturbation, subject to the constraints that (1) only a subset of entries may be perturbed (resource allocations are only permitted at some network locations); and (2) the sum of the perturbed entries is zero (resource re-distribution) or fixed (allocation on a fixed resource budget). The aim of the design is to optimize the dominant eigenvalue of the Metzler matrix, which captures or approximates a dominant propagative dynamics in the network. Succinctly, the problem addressed here is the design of trace-preserving or fixed-trace diagonal perturbations of an essentially nonnegative matrix to minimize a dominant eigenvalue, in the case where only a subset of entries can be designed. We study this *fixed-trace* ***subset design problem***, with a focus on developing graph-theoretic insights into the optimal solution and addressing resource re-design when constraints are changed.

This study extends a research effort in the linear-algebra literature on optimizing the dominant eigenvalue of an essentially-nonnegative matrix over trace-preserving or fixed-trace diagonal perturbations (Johnson, Loewy, Olesky, & Van Den Driessche, 1996; Johnson, Stanford, Dale Olesky, & van den Driessche, 1994), which is part of a broader effort on the fast eigen-decomposition of these matrices (see Johnson, Pitkin, and Stanford (2000), Schneider and Zenios (1990), Zhang, Qi, Luo, and Xu (2013)). These works

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exploit the convexity of the dominant eigenvalue with respect to the diagonal entries along with a similarity transformation to a *line-sum-symmetric* form (where each row sum is equal to the corresponding column sum), to develop computationally-appealing solutions and some structural insights into the optimization (Eaves, Hoffman, Rothblum, & Schneider, 1985; Johnson et al., 1994). This study also contributes to a thrust on resource-constrained control of spread dynamics in the controls community (Enyioha, Preciado, & Pappas, 2013; Preciado, Zargham, Enyioha, Jadbabaie, & Pappas, 2014; Ramirez-Llanos & Martinez, 2014; Ramírez-Llanos & Martínez, 2015; Robertson, Eisenberg, & Tien, 2013; Wan, Roy, & Saberi, 2008), which has addressed parallel optimization problems and generalizations to those considered in the linear-algebra literature, using both structural and numerical approaches (Preciado et al., 2014; Ramírez-Llanos & Martínez, 2015). Of particular relevance, algebraic characterizations of the optimum and numerical optimization algorithms were developed for the subset-design problem in Abad Torres, Roy, and Wan (2017, 2015). The presented research also contributes to a growing effort to characterize the input–output dynamics of sparsely actuated and measured network dynamics (Abad Torres & Roy, 2015b, c; Dhal & Roy, 2013; Liu, Slotine, & Barabási, 2013; Pasqualetti, Zampieri, & Bullo, 2014; Rahmani, Ji, Mesbahi, & Egerstedt, 2009; Roy, Xue, & Das, 2012; Xue, Wang, & Roy, 2014).

Relative to the literature, the main contribution of this study is to (1) develop graph-theoretic insights into the optimal subset design and its performance and (2) systematically address resource re-design as constraints are changed. In particular, we show that the pattern of resource distribution at the optimum is closely tied to the network's graph (the pattern of zero and nonzero entries of the matrix) and the locations of control channels (or designable resources) relative to the graph. Algorithms for resource re-design are also obtained, and the re-allocation is shown to be specially patterned for certain network structures. As a whole, the study shows how resource placements can account for the undesignable structure of a network in shaping response characteristics. We note that some results apply to arbitrary essentially-nonnegative matrices, while are specialized to particular sub-classes (e.g., diagonally symmetrizable, line-sum symmetric, or having a special graph structure).

The article is organized as follows. The design problem is introduced in Section 2. Preliminary algebraic analyses and design algorithms are reviewed in Section 3. Graph-theoretic results on the optimal design are described in Section 4, and the re-design problem is addressed in Section 5. An example is presented (Section 6), and brief conclusions are given (Section 7). Initial results in this direction were given in Abad Torres and Roy (2015a).

2. Problem formulation and notation

An $n \times n$ real essentially-nonnegative (or Metzler) matrix A is considered. The problem of interest is to find a fixed-trace diagonal perturbation matrix $D = \text{diag}(D_1, \ldots, D_n)$ such that the dominant eigenvalue of $A + D$ is minimized, subject to the further constraint that some entries of D are restricted to be zero (say, $D_i = 0$ for $i = m+1, \ldots, n$, without loss of generality). This problem can be formalized as follows:

$$\begin{aligned} &\underset{D_1,\ldots,D_m}{\text{argmin}} \ \lambda_{\max}(A+D) \\ &\quad \text{s.t. } D_i = 0 \ \ \forall i = m+1, \ldots, n, \\ &\quad \sum_{i=1}^{m} D_i = \Gamma, \end{aligned} \tag{1}$$

where Γ specifies the trace of the imposed perturbation, and $\lambda_{\max}$ refers to the dominant eigenvalue, i.e. the eigenvalue whose real part is largest (most positive). Since $A + D$ is essentially nonnegative, this dominant eigenvalue is real, see Cohen (1981). Some results are focused specifically on the trace-preserving case, where $\Gamma = 0$.

The problem can be interpreted as a resource allocation task, where finite resources D_i are being placed at a subset of network locations to suppress a linear propagative dynamics governed by the state matrix A (with more negative D_i corresponding to higher resource levels). For such network applications, the zero–nonzero pattern of the matrix A specifies the network's topology. Thus, to enable graph-theoretic analysis, we associate with the matrix A a weighted digraph $\mathcal{G} = (V, E : W)$, where the vertices contained in V are labeled $1, \ldots, n$, an arc (directed edge) is drawn from vertex i to vertex j ($i \neq j$) if and only if $A_{j,i} \neq 0$, and the arc is assigned a weight $A_{j,i}$.

Some matrix and graph terminology/notation is used in our development. The entries in D that are not constrained to be zero (and corresponding graph vertices) are termed *designable entries (vertices)*; the constrained entries/vertices are called *undesignable*. The diagonal matrix D that minimizes the dominant eigenvalue of $A+D$ is denoted as $\bar{D}$. The dominant eigenvalue and corresponding eigenvectors of $A + \bar{D}$ are denoted as $\bar{\lambda}_{\max}$, $\bar{w}_{\max}$ and $\bar{v}_{\max}$. Further, $w_{\max,i}$ and $v_{\max,i}$ refer to the ith entries of left- and right-eigenvectors associated with the dominant eigenvalue. A couple of standard graph-theoretic terms are also used: a vertex cut set is a set of vertices whose removal results in a disconnected graph, while an (edge) cut set is a set of edges whose removal results in a disconnected graph.

3. Preliminaries: algebraic analysis and algorithms

In Abad Torres et al. (2017), an algebraic analysis was conducted of the spectrum of $A + D$ for the optimal fixed trace subset perturbation design $D = \bar{D}$, and used to develop an algorithm for finding the optimal perturbation. These analyses, which are preliminary to the results developed here, are reviewed (without proof) in the following theorem and lemma.

Theorem 1. *Consider the matrix $A+D$, where $D = diag(D_1, \ldots, D_m, 0, \ldots, 0)$ and A is a real essentially-nonnegative matrix (which may or may not be irreducible). Consider any $D = \bar{D}$ that minimizes the dominant eigenvalue of $A + D$ subject to $\sum_{i=1}^{m} D_i = \Gamma$. Assume that $A + \bar{D}$ has a real simple dominant eigenvalue. The left and right dominant eigenvectors, $\bar{w}_{\max}$ and $\bar{v}_{\max}$, of $A + \bar{D}$ satisfy one of the following conditions: (1) There exists $\bar{\mu} > 0$ such that $\bar{w}_{\max,i}\bar{v}_{\max,i} = \bar{\mu} \quad \forall i = 1, 2, \ldots, m$; (2) $\bar{w}_{\max,i}\bar{v}_{\max,i} = 0 \quad \forall i = 1, 2, \ldots, m$. Furthermore, if A is irreducible, then $A+\bar{D}$ has a real simple dominant eigenvalue, and the optimizing $\bar{D}$ and the dominant eigenvectors always satisfy condition 1.*

The algorithm for computing the optimal trace-preserving diagonal perturbation matrix requires some further notation. Specifically, it is useful to partition the *topology matrix* A as $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$, where A_{11} is an $m \times m$ matrix. The result also draws on the fact that there always exists a diagonal similarity transformation matrix P such that $PAP^{-1}\vec{1} = P^{-1}A'P\vec{1}$, where $\vec{1}$ is the all ones vector of the appropriate dimension and A' is the transpose of A (see Eaves et al. (1985), Schneider and Zenios (1990) for the computation of the row-sum-symmetrizing transformation P). Here is the algorithm:

Lemma 1. *Consider the matrix $A + D$, where $D = diag(D_1, \ldots, D_m, 0, \ldots, 0)$, and A is an irreducible essentially-nonnegative matrix. The*