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Stubborn state observers for linear time-invariant systems ☆

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ABSTRACT

For the purpose of estimating the state of a linear time-invariant system with measurements subject to outliers, we propose an observer with a saturated output injection in such a way to mitigate the effect of abnormal and isolated measurement noise on the error dynamics. Stability conditions in both the continuous-time and the discrete-time cases are derived, which ensure global exponential stability to the origin for the error dynamics. Such conditions can be expressed in terms of linear matrix inequalities, allowing for a viable design by using convex optimization. The effectiveness of the approach is illustrated by means of simulations in comparison with the Luenberger observer.

1. Introduction

The long-standing problem of estimating the state variable of a plant was brilliantly solved in 1964 by Luenberger, who proposed in Luenberger (1964) to use a dynamic setup, called *observer*, which provides asymptotic state estimates under the action of an output injection (see also Luenberger, 1966 for a further development). This pioneering result was extended from linear time-invariant (LTI) continuous-time systems to nonlinear and/or discrete-time systems with a huge, still growing literature. The observers for LTI systems have a linear structure, which is well-suited for verifying the asymptotic stability of the estimation error. In this paper, we focus on observers for continuous-time and discrete-time LTI plants with a nonlinear output injection because of a *saturation* that depends on a variable threshold. The reason why we propose this novel estimation paradigm is that we aim at reducing the effect of measurement outliers, i.e., impulsive disturbances that may irremediably corrupt the measurements used for the purpose of state estimation.

The literature on estimation in the presence of outliers is vast but it refers mainly to identification problems. Most of the results rely on the idea of setting the Kalman filter so as to make it robust to outliers (see, among others, De Palma and Indiveri, 2017; Gandhi and Mili, 2010; Shi, Chen, and Shi, 2013). In Akkaya and Tiku (2008) statistical tests are proposed that are less sensitive to abnormal noises. Identification based on an l_1 criterion is addressed in Lauer, Bloch, and Vidal (2011) and Xu, Bai, and Cho (2014). Instead of only attenuating the effect of outliers, a different approach is reported in Alessandri and Awawdeh (2016), where a method based on a leave-one-out moving-horizon estimation strategy is proposed.

In the past two decades suitable characterizations of input saturation in control systems have been developed, allowing to reach out beyond the mere application of absolute stability concepts and global sector properties of the saturation nonlinearity (Khalil, 2002). In particular, according to the results in Sontag (1984), it was recognized that global exponential stabilization of a linear plant using a bounded input is impossible, unless the plant is already exponentially stable. Therefore, suitable generalizations of the standard (globally-based) absolute stability results were proposed by Gomes da Silva Jr and Tarbouriech (2005), Hu, Lin, and Chen (2002) and Hu, Teel, and Zaccarian (2008), which allowed to establish local results with a guaranteed region of attraction by way of a generalized (or local/regional) sector condition satisfied by the saturation nonlinearity (see also Hu and Lin, 2001 and Tarbouriech, Garcia, Gomes da Silva Jr., and Queinnec, 2011). This generalized sector condition is nowadays a well known tool to address analysis and design of control laws for linear systems subject to saturations and deadzones.

In line with Alessandri and Zaccarian (2015), here we employ a saturation function in the output injection signal of a linear asymptotic observer for a linear plant. As compared to the typical use of saturations, the novelty consists in using the saturation level artificially and adjusting it in such a way that global asymptotic stability

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properties of the error dynamics are guaranteed for any detectable linear plant. The adaptive saturation of the output error makes it possible to deal with specific types of measurement disturbances, such as the ones characterized by rare outliers. We call the resulting observers *stubborn* because indeed the saturation threshold is guaranteed to converge to zero in the absence of noise, and then possible outliers do not directly reach the error dynamics since they are mitigated by the limiting effect of saturation. On the other hand, persistent estimation errors gradually cause an increase of the saturation threshold and become increasingly important in the error dynamics, so as to guarantee global asymptotic stability of the origin for the estimation error dynamics. Interestingly, for the continuous-time case we can strengthen the global asymptotic stability (GAS) results to global exponential stability (GES) of the origin. This is however not the case for the parallel discrete-time results, where a significantly different proof technique must be adopted and the weaker GAS property of the origin can be established. We provide convex conditions for the selection of the observer parameters that can be expressed by means of linear matrix inequalities (LMIs) and easily solved by using semidefinite programming (SDP) tools (Boyd, El Ghaoui, Feron, & Balakrishnan, 1994). Moreover, we prove that such conditions are always feasible under the (necessary) assumption that the plant state is detectable from the available measurement output. Finally, it is worth mentioning that there exist some results on saturated high-gain observers for nonlinear systems. In Lei, Wei, and Li (2007), the saturation is used to ensure that the estimate takes into account the knowledge on the boundedness of the state trajectories. A nested-saturation low-power approach is proposed in Astolfi, Marconi, and Teel (2016) and Teel (2016) to overcome the well-known issue of the peaking, which affects high-gain observers.

Preliminary results of this work were presented in Alessandri and Zaccarian (2015) only for the continuous-time case with a first proposal of tuning procedure for the observer parameters. Here, in addition to extending the results to the discrete-time case, we provide the proofs that were missing in Alessandri and Zaccarian (2015), we discuss a novel tuning procedure for the observer parameters, and we illustrate the effectiveness of the proposed strategy on new simulation studies. We note that the discrete-time case discussed here presents somewhat different challenges for the proof of stability, which must follow a different paradigm.

The paper is structured as follows. The main results concerning the proposed stubborn estimator in the continuous-time and discrete-time cases are presented in Sections 2 and 3, respectively. Since numerical results in the first case are reported in Alessandri and Zaccarian (2015), novel simulation results concerning only the second case for discrete-time LTI systems are shown in Section 4. Based on the foregoing, conclusions are drawn in Section 5.

Notation. The minimum and maximum eigenvalues of a real, symmetric matrix P are denoted by $\lambda_{\min}(P)$ and $\lambda_{\max}(P)$, respectively; in addition, $P > 0$, $P \geq 0$, $P < 0$, $P \leq 0$, means that P is positive definite, positive semidefinite, negative definite, and negative semidefinite, respectively. Given a generic matrix M, $\mathrm{He}\, M := M^\top + M$ and $|M| := \left(\lambda_{\max}(M^\top M)\right)^{1/2} = \left(\lambda_{\max}(MM^\top)\right)^{1/2}$. Finally, let $(x, y) := [x^\top, y^\top]^\top$, where x and y are two vectors.

2. Main results for continuous-time LTI systems

2.1. Observer architecture

We consider an LTI continuous-time plant with state $x \in \mathbb{R}^n$, output $y \in \mathbb{R}^{n_y}$, control input $u \in \mathbb{R}^{n_u}$, and disturbance $d \in \mathbb{R}^{n_d}$:

$$\begin{aligned} \dot{x} &= Ax + B_u u + B_d d \\ y &= Cx + D_u u + D_d d. \end{aligned} \tag{1}$$

In this paper, we propose the following dynamic state estimator:

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + B_u u + L\,\mathrm{sat}_\sigma(y - \hat{y}) \\ \hat{y} &= C\hat{x} + D_u u, \end{aligned} \tag{2a}$$

where sat_σ is a symmetric vector saturation function with variable non-negative saturation limits $\sigma = (\sigma_1, \ldots, \sigma_{n_y}) \in \mathbb{R}^{n_y}_{\geq 0}$ defined for each $s = (s_1, \ldots, s_{n_y}) \in \mathbb{R}^{n_y}$ as

$$\mathrm{sat}_\sigma(s) = \begin{bmatrix} \mathrm{sat}_{\sigma_1}(s_1) \\ \vdots \\ \mathrm{sat}_{\sigma_{n_y}}(s_{n_y}) \end{bmatrix}, \tag{2b}$$

$\mathrm{sat}_{\sigma_k}(s_k) = \max\{-\sigma_k, \min\{\sigma_k, s_k\}\}$ being the standard scalar symmetric saturation function. The observer dynamics is completed by the following scalar equation governing the evolution of the saturation limits σ:

$$\dot{\overline{\sigma}} = -\lambda\overline{\sigma} + (y - \hat{y})^\top R (y - \hat{y}), \quad \overline{\sigma} \in \mathbb{R}_{\geq 0} \tag{2c}$$

$$\sigma_i = \sqrt{\overline{\sigma}/w_i}, \quad i = 1, \ldots, n_y, \tag{2d}$$

where $\lambda > 0$ scalar, $R = R^\top > 0$ and $w_i > 0$, $i = 1, \ldots, n_y$ are suitable parameters. Moreover, by $\overline{\sigma} \in \mathbb{R}_{\geq 0}$ we mean that the dynamics of $\overline{\sigma}$ is constrained to the non-negative real axis, which is indeed forward invariant for dynamics (2c). This constraint clearly makes the square root in (2d) well posed.

The rationale behind the nonlinear observer (2) is that of a "stubborn" observer, in the sense that the output estimation error $e_y := y - \hat{y} \in \mathbb{R}^{n_y}$ is injected in the observer equations (2a) through a bounded function. The advantage of such an observer structure is that possible outliers in the measurements are suitably canceled (actually saturated) and their effect on the estimation error is reduced. Dynamic adaptation of the saturation level is then necessary in such a way that global asymptotic stability of the estimation error dynamics can be ensured. In particular, in the adaptation Eq. (2c), the effect of the first term (involving λ) is to push to zero the saturation levels σ, while the effect of the second term (depending on R) is to ensure that an output estimation error eventually leads to a suitable increase of σ in such a way that the error dynamics can be stabilized.

2.2. Stability and feasibility

Given the observer architecture (2), we provide here necessary conditions on the observer parameters L, λ, R, and W that ensure uniform global asymptotic stability of the dynamics of the error variables $(e, \overline{\sigma})$, where $e := x - \hat{x}$, in the absence of disturbances, i.e., with $d = 0$. This dynamics can be easily computed from (1) and (2) as follows

$$\begin{aligned} \dot{e} &= (A - LC)e + Lq \\ \dot{\overline{\sigma}} &= -\lambda\overline{\sigma} + e^\top C^\top RCe, \quad \overline{\sigma} \in \mathbb{R}_{\geq 0} \\ q &= \mathrm{dz}_\sigma(Ce) := Ce - \mathrm{sat}_\sigma(Ce), \end{aligned} \tag{3}$$

where the deadzone function $\mathrm{dz}_\sigma(s) := s - \mathrm{sat}_\sigma(s)$ and the auxiliary variable $q \in \mathbb{R}^{n_y}$ are introduced to simplify the notation.

Remark 1. Note that the adaptation law in (2c) is necessary if one wants to propose an observer with a saturated output injection for general linear plants. Indeed, the simple alternative of selecting a constant (or bounded) saturation level σ would dramatically fail to work whenever the matrix A has eigenvalues with positive real part. This is apparent if one realizes that the error dynamics (3) corresponds to a linear plant with a saturated linear output injection and, as established in Sontag (1984), exponentially unstable linear dynamics cannot be globally asymptotically stabilized through a bounded input. In this paper we propose the dynamic saturation level in (2c), (2d) which instead makes our solution applicable to any linear plant, as long as the pair (C, A) is detectable, as established later in Proposition 1.