Automatica 89 (2018) 201–211

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Chattering in the Reach Control Problem ☆

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ARTICLE INFO

Article history:
Received 17 January 2017
Received in revised form 20 September 2017
Accepted 23 October 2017

Keywords:
Reach Control Problem
Piecewise affine feedback
Switched control
Zeno behaviour

ABSTRACT

The Reach Control Problem (RCP) is a fundamental problem in hybrid control theory. The goal of the RCP is to find a feedback control that drives the state trajectories of an affine system to leave a polytope through a predetermined exit facet. In the current literature, the notion of leaving a polytope through a facet has an ambiguous definition. There are two different notions. In one, at the last time instance when the trajectory is inside the polytope, it must also be inside the exit facet. In the other, the trajectory is required to cross from the polytope into the outer open half-space bounded by the exit facet. In this paper, we provide a counterexample showing that these definitions are not equivalent for general continuous or smooth state feedback. On the other hand, we prove that analyticity of the feedback control is a sufficient condition for equivalence of these definitions. We generalize this result to several other classes of feedback control previously investigated in the RCP literature, most notably piecewise affine feedback. Additionally, we clarify or complete a number of previous results on the exit behaviour of trajectories in the RCP.

1. Introduction

In the past period, there has been a significant effort to formalize the mathematical foundations of switched and hybrid control systems. Due to the discontinuous nature of such systems, fundamental results guaranteeing the existence and uniqueness of solutions of classical ODEs with continuous vector fields no longer hold automatically. A new theory of existence and uniqueness of solutions of switched and hybrid systems has been formulated by, among others, Heemels, Çamlibel, van der Schaft, and Schumacher (2002), Imura and van der Schaft (2000) and Lygeros, Johansson, Simić, Zhang, and Sastry (2003).

An additional property specific to switched and hybrid systems is *Zeno* behaviour, in which a trajectory, even if guaranteed to exist and be unique, undergoes an infinite number of switches, i.e., discontinuous changes in the governing vector field, in a finite time interval. This property has been the subject of intense recent research, e.g., by Ames and Sastry (2005), Çamlibel (2008), Goebel and Teel (2008) and Heymann, Lin, Meyer, and Resmerita (2005).

Additionally, a number of classical control notions such as controllability, observability, and Lyapunov stability do not apply to systems governed by discontinuous vector fields. There has been significant work to extend these concepts to switched and hybrid systems (see, e.g., Branicky, 1998; Ezzine & Haddad, 1988; Petterson & Lennartson, 1996). For a comprehensive treatment of discontinuous dynamical systems see the work by Cortés (2008) and Liberzon (2003).

This paper follows the above line of research on deepening the mathematical foundations of hybrid control system, but here we focus on reach control theory. The central problem in this theory is the Reach Control Problem (RCP) (Habets, Collins, & van Schuppen, 2006; Roszak & Broucke, 2006). Further work appeared in Ashford and Broucke (2013), Belta, Habets, and Kumar (2002), Broucke (2010), Broucke and Ganness (2014), Habets and van Schuppen (2004), Helwa and Broucke (2013, 2014), Helwa and Broucke (2015), Helwa, Lin, and Broucke (2016), Moarref, Ornik, and Broucke (2016), Ornik and Broucke (2017), Semsar-Kazerooni and Broucke (2014) and Wu and Shen (2016). The goal of the RCP is to find a feedback control u such that, for any initial state x_0 inside a polytope $\mathcal{P}$, the trajectory $\phi(\cdot, x_0)$ of an affine control system $\dot{x} = Ax + Bu + a$ leaves $\mathcal{P}$ through a predetermined exit facet $\mathcal{F}_0$ in finite time, without first leaving $\mathcal{P}$ through any other facets. While there is an extensive literature on reach control theory, this is the first paper that focuses solely on a formal and complete discussion of existence, uniqueness, and behaviour of solutions.

The intention is for the RCP to serve as a building block in a hybrid control strategy that rests upon triangulating the state space to achieve some control objective. For example, if the system state is desired to go from one area of the state space to another, this can be achieved by partitioning the entire state space into simplices or polytopes, and constructing a sequence of polytopes

☆ The material in this paper was not presented at any conference. This paper was recommended for publication in revised form by Associate Editor Denis Arzelier under the direction of Editor Richard Middleton.

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https://doi.org/10.1016/j.automatica.2017.11.008
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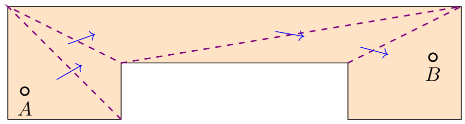

Fig. 1. An example of a reach control approach to solving a control problem. The state space is given in red, and the control objective is to guide the system state from point A on the left to point B on the right. The state space is cut into polytopes, and the goal is to define a controller on each polytope such that the desired sequences of polytopes (denoted by blue arrows) is followed. The exit facets of all polytopes in sequence are marked in purple. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

such that the state trajectories move through the polytopes in the desired order, until they finally reach the last polytope (see Fig. 1).

Numerous applications of the RCP have already been identified. These include biomolecular networks (Belta et al., 2002), control of aircraft (Belta & Habets, 2006), process control (Haugwitz & Hagander, 2007), aggressive manoeuvres of mechanical systems (Vukosavljev & Broucke, 2014), quadcopter motion (Vukosavljev, Jansen, Broucke, & Schoellig, 2016), and automatic parallel parking of vehicles (Ornik, Moarref, & Broucke, 2017).

This paper focuses on what happens when trajectories transition from one polytope to the next. In order to make the transitions between polytopes work, it is not only necessary for a trajectory to exit a polytope $\mathcal{P}$ with its last point in $\mathcal{P}$ lying on the desired exit facet $\mathcal{F}_0$. We must ensure that this exit will simultaneously result in the trajectory entering the next polytope in the desired sequence. This paper investigates a fundamental question in reach control theory which has not been addressed by previous work on general hybrid or switched systems: *what is the appropriate notion of leaving a polytope or a simplex through a facet?*

In Habets and van Schuppen (2001, 2004), it was required that velocity vectors must point strictly outside the polytope at points in the exit facet. This condition implies that a trajectory arriving at the exit facet will immediately enter the open half-space outside $\mathcal{P}$ and bounded by the exit facet. Sufficient conditions were given in Habets and van Schuppen (2004) for a Lipschitz continuous feedback to solve this problem. The proof assumes strict inward or outward conditions on velocity vectors along the facets of the polytope. When these conditions are not strict, certain pathologies can arise, as this paper will show, and arguments about whether trajectories lie in certain half-spaces with respect to facets are considerably more delicate. Lemma 3 of Roszak and Broucke (2006) regards trajectories exiting a polytope via a facet but without necessarily crossing into the outer open half-space. In Section 3.4, we provide a complete proof of a stronger version of this result.

One goal of this paper is to explore the relationship between the two notions for exiting a polytope. We consider the following question: *Is it possible for a trajectory to leave a polytope $\mathcal{P}$ but without crossing into an outer half-space?* When a trajectory exits $\mathcal{P}$ but does not cross into an outer half-space, we say it *chatters*. A second goal of the paper is to identify appropriate classes of feedback controls that do not allow chattering.

The paper is organized as follows. In Section 2, we define the Reach Control Problem and discuss the nuanced notions of *exiting through* a facet, *crossing* a facet, and *chattering*. In Section 3, we explore conditions on the vector field to disallow chattering. Section 3.1 discusses chattering under various feedback classes previously studied for the RCP. Section 3.2 focuses on the important class of continuous piecewise affine feedbacks. In Section 3.3, we apply these results to the Output Reach Control Problem (ORCP), first studied in Kroeze and Broucke (2016). Section 3.4 further discusses affine feedback control. Finally, Section 4 explores

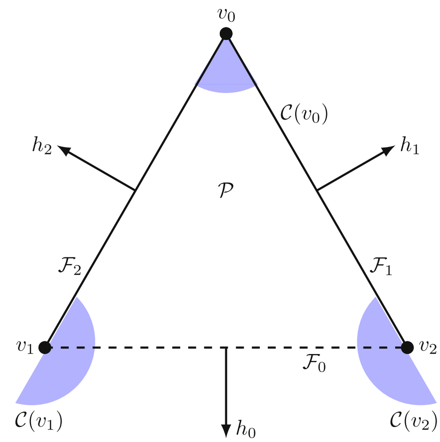

Fig. 2. An illustration of notation used in the paper. The polytope $\mathcal{P} = \mathrm{co}\{v_0, v_1, v_2\}$ is given by vertices $V_{\mathcal{S}} = \{v_0, v_1, v_2\}$ and facets $\mathcal{F}_0$, $\mathcal{F}_1$, and $\mathcal{F}_2$, with each facet indexed by the vertex it does not contain. h_i is the unit normal vector pointing out of $\mathcal{S}$. $\mathcal{F}_0$ is designated as the exit facet. Because of their previously discussed geometric meaning, the cones $\mathcal{C}(v_i)$ are illustrated attached at each v_i. However, by (2), each cone $\mathcal{C}(x)$ has its apex at 0.

discontinuous piecewise affine feedback, as developed in Broucke and Ganness (2014).

Notation. Let $\mathcal{K} \subset \mathbb{R}^n$ be a set. The complement of $\mathcal{K}$ is $\mathcal{K}^c := \mathbb{R}^n \backslash \mathcal{K}$, and the set difference of two sets $\mathcal{K}_1, \mathcal{K}_2 \subset \mathbb{R}^n$ is denoted by $\mathcal{K}_1 \backslash \mathcal{K}_2$. The closure of set $\mathcal{K}$ is $\overline{\mathcal{K}}$. For two vectors $x, y \in \mathbb{R}^n$, $x \cdot y$ denotes the inner product of the two vectors. The notation $\mathrm{co}\{v_1, v_2, \ldots\}$ denotes the convex hull of a set of points $v_i \in \mathbb{R}^n$, and $\mathrm{aff}(\mathcal{K})$ is the affine hull of set $\mathcal{K}$.

2. Problem statement

Consider an n-dimensional polytope $\mathcal{P} := \mathrm{co}\{v_0, \ldots, v_p\}$ with vertex set $V := \{v_0, \ldots, v_p\}$. A *facet* of $\mathcal{P}$ is an $(n-1)$-dimensional face of $\mathcal{P}$. Let $\mathcal{F}_0, \mathcal{F}_1, \ldots, \mathcal{F}_r$ denote the facets of $\mathcal{P}$. The facet $\mathcal{F}_0$ is referred to as the *exit facet*, while $\mathcal{F}_1, \ldots, \mathcal{F}_r$ are called *restricted facets*. Let $J = \{1, \ldots, r\}$ and let h_i be the unit normal to each facet $\mathcal{F}_i$ pointing outside the polytope. We note that each point on the boundary of $\mathcal{P}$ can belong to one or more facets. An example is given in Fig. 2, where vertices of $\mathcal{P}$ belong to two facets, and other points on the boundary of $\mathcal{P}$ to one.

We consider the affine control system defined on $\mathcal{P}$:

$$\dot{x} = Ax + Bu + a, \tag{1}$$

where $A \in \mathbb{R}^{n \times n}$, $a \in \mathbb{R}^n$, $B \in \mathbb{R}^{n \times m}$, and $\mathrm{rank}(B) = m$. Let $\mathcal{B} = \mathrm{Im}(B)$, the image of B. Let $\phi(\cdot, x_0)$ denote the trajectory of (1) under some control law u starting from $x_0 \in \mathcal{P}$. The standard formulation of the RCP is as follows (Habets et al., 2006; Roszak & Broucke, 2006).

Problem 1 (*Reach Control Problem (RCP)*). Consider system (1) defined on $\mathcal{P}$. Find a map $u : \mathcal{P} \to \mathbb{R}^m$ such that for every $x_0 \in \mathcal{P}$, there exist $T \geq 0$ and $\varepsilon > 0$ such that

(i) $\phi(t, x_0) \in \mathcal{P}$ for all $t \in [0, T]$,
(ii) $\phi(T, x_0) \in \mathcal{F}_0$, and
(iii) $\phi(t, x_0) \notin \mathcal{P}$ for all $t \in (T, T + \varepsilon)$.

We emphasize that the current setting of the RCP as given in Problem 1 does not stipulate that a system trajectory should leave $\mathcal{P}$ immediately after first entering the exit facet $\mathcal{F}_0$. Indeed, it is allowed for a system trajectory to touch the exit facet $\mathcal{F}_0$ and then go back into $\mathcal{P}$ before leaving through $\mathcal{F}_0$ at some later point.