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Brief paper

# Some insights into the migration of double imaginary roots under small deviation of two parameters[☆]

Dina Alina Irofti [a,*], Keqin Gu [b], Islam Boussaada [a], Silviu-Iulian Niculescu [a]

[a] Laboratoire des Signaux et Systèmes (L2S) CNRS-CentraleSupélec-Université Paris-Sud, 3 rue Joliot-Curie, 91192 Gif-sur-Yvette cedex, France
[b] Department of Mechanical and Industrial Engineering, Southern Illinois University Edwardsville, Edwardsville, IL 62026, USA

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ABSTRACT

This paper studies the migration of double imaginary roots of the systems' characteristic equation when two parameters are subjected to small deviations. The proposed approach covers a wide range of models. Under the least degeneracy assumptions, we found that the local stability crossing curve has a cusp at the point that corresponds to the double root, and it divides the neighborhood of this point into an S-sector and a G-sector. When the parameters move into the G-sector, one of the roots moves to the right half-plane, and the other moves to the left half-plane. When the parameters move into the S-sector, both roots move either to the left half-plane or the right half-plane depending on the sign of a quantity that depends on the characteristic function and its derivatives up to the third order.



## 1. Introduction

Control systems often depend on parameters and we may generally write their characteristic equation as

$$q(s, p) = 0, \tag{1}$$

where $s$ is the Laplace variable and $p \in \mathbb{R}^n$ is a vector of $n$ parameters. We can have parameters due to internal dynamics. For instance, modeling in physical, biological or social sciences sometimes requires taking into account the time delays inherent in the phenomena. Depending on the model complexity, but also on how much information is known, we may chose a model with continuous constant delays, or a model with distributed delays (see Cushing, 1977; MacDonald, 1989). For instance, in the case of a time-delay system with two constant delays, the characteristic equation can be written of the form

$$q_1(s, \tau_1, \tau_2) = r_0(s) + r_1(s)e^{-\tau_1 s} + r_2(s)e^{-\tau_2 s}, \tag{2}$$

where $r_k(s)$, $k = 0, 1, 2$ are polynomials of $s$ with real coefficients, and the delays $\tau_1$, $\tau_2$ are the two parameters.

Also common is the case when $p$ contains controller parameters. Classical examples include PI, PD and PID controllers. For example, the continuous time PID controller is expressed in the Laplace domain as $q_2(s) = K_P\left(1 + \frac{1}{T_i s} + T_d s\right)$, where $K_P$ is the proportional gain, $T_i$ and $T_d$ are the integral and derivative time constants. Furthermore, many process control problems also contain a time delay $\tau_m$ (see Morarescu, Mendez-Barrios, Niculescu, & Gu, 2011; O'Dwyer, 2006). These include proportional plus delay $q_3(s)$, integrator plus delay model $q_4(s)$, first order lag plus delay $q_5(s)$, first order lag plus integral plus delay $q_6(s)$ expressed below:

$$q_3(s) = K_m(1 + e^{-s\tau_m}) \qquad q_4(s) = \frac{K_m e^{-s\tau_m}}{s}$$
$$q_5(s) = \frac{K_m e^{-s\tau_m}}{1 + sT_m} \qquad q_6(s) = \frac{K_m e^{-s\tau_m}}{s(1 + sT_m)}.$$

If in the expression of $q_3(s)$ there are two different gains for the two terms, then we obtain the *proportional retarded* controller: $q_7(s) = K_p + K_r e^{-s\tau_m}$. Furthermore, Villafuerte, Mondié, and Garrido (2013) showed that proportional retarded controller outperforms a PD controller on an experimental DC-servomotor setup. Obviously, any control among PID type results in a characteristic equation that depends on the control parameters.

Many studies have been conducted on the stability of systems that depend on parameters. For example, for systems with a single delay as the parameter, methods of identifying all the stable delay intervals are given in Lee and Hsu (1969) and Walton and Marshall (1987). For system with two parameters, a

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* Corresponding author.
*E-mail addresses:* Dina.Irofti@l2s.centralesupelec.fr (D.A. Irofti), kgu@siue.edu (K. Gu), Islam.Boussaada@l2s.centralesupelec.fr (I. Boussaada), Silviu.Niculescu@l2s.centralesupelec.fr (S.-I. Niculescu).

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rich collection of stability charts (the parameter regions showing where the system is stable) for time-delay systems are presented in Stépán (1989). For systems with two delays as the parameters, a geometric approach is introduced in Gu, Niculescu, and Chen (2005). This analysis is based on the continuity of the characteristic roots as functions of parameters, which needs to be carefully evaluated in the case of time delay systems of neutral type (see Gu, 2012; Michiels & Niculescu, 2014), and consists of identifying the parameters that correspond to imaginary characteristic roots and judging the direction of crossing of these roots as parameters change. Such an analysis is known as D-decomposition method (also known as D-subdivision method). Such a method is first applied to time-delay systems in Neimark (1948). Earlier examples for other systems have been documented by the survey paper (Gryazina, Polyak, & Tremba, 2008). More recent studies using D-decomposition methods can be found in Gryazina and Polyak (2006) and Ruan and Wei (2003). Challenges due to non-differentiability arise when the imaginary roots are also multiple roots. Such problems have traditionally been solved using Puiseux series (Kato, 1980; Knopp, 1996), see, for example, Chen, Fu, Niculescu, and Guan (2010) and Li, Niculescu, Çela, Wang, and Cai (2013) for systems with one parameter.

In this paper, we study systems with two parameters, and present a method to analyze the migration of roots in a neighborhood of the parameters corresponding to a double imaginary characteristic root. The method of analysis uses traditional complex analysis, and does not require Puiseux series. A preliminary version of this paper, which is restricted to the case of two point-wise delays as the parameters, was presented in Gu, Irofti, Boussaada, and Niculescu (2015). It should be pointed out that some phenomena discussed in this work, such as cusp in the parameter space, has also been presented in Levantovskii (1982). In this paper, we extend and generalize this method to a wide range of systems, as mentioned above, that can generally be written in the form of characteristic equation (1). Additionally, we illustrate how to apply the algebraic criterion by three examples.

## 2. Problem statement and prerequisites

Consider a system with the characteristic equation of the form (1). For $p_0 = (p_{10}, p_{20})$, we assume that the function $q(s, p_0)$ has a double root on the imaginary axis, $s = s_0 = i\omega_0$. In other words, we assume

$$q(s_0, p_0) = \left.\frac{\partial q}{\partial s}\right|_{\substack{s=s_0\\ p=p_0}} = 0. \tag{3}$$

We further assume that $s_0$ is not a third order root, i.e.

$$\left.\frac{\partial^2 q}{\partial s^2}\right|_{\substack{s=s_0\\ p=p_0}} \neq 0. \tag{4}$$

Suppose $q(s, p)$ is analytic with respect to $s$, and continuously differentiable with respect to $(s, p)$ up to the third order. We make the following additional non-degeneracy assumption:

$$D = \det \begin{pmatrix} \mathrm{Re}\left(\dfrac{\partial q}{\partial p_1}\right) & \mathrm{Re}\left(\dfrac{\partial q}{\partial p_2}\right) \\ \mathrm{Im}\left(\dfrac{\partial q}{\partial p_1}\right) & \mathrm{Im}\left(\dfrac{\partial q}{\partial p_2}\right) \end{pmatrix}_{\substack{s=s_0\\ p_1=p_{10}\\ p_2=p_{20}}} \neq 0, \tag{5}$$

where $\mathrm{Re}(\cdot)$ and $\mathrm{Im}(\cdot)$ denote the real and imaginary part of a complex number, respectively. Eqs. (3)–(5) will be the standing assumptions in the remaining part of this paper. Assumption (5) contains the first-order partial derivatives of $q$ with respect to the two parameters, $p_1$ and $p_2$.

**Definition 1.** For a system of the form (1) that satisfies (3), we say it is "the least degenerate" if assumptions (4)–(5) hold. We also say that inequalities (4) and (5) are the least degeneracy assumptions.

In view of the implicit function theorem, a consequence of inequality (5), which is one of the non-degeneracy assumptions, is that the characteristic equation (1) defines the pairs $(p_1 \;\; p_2)$ in a small neighborhood of the critical point $p_0 = (p_{10} \;\; p_{20})$ as a function of $s$ in a sufficiently small neighborhood of $s_0$. Introduce the notation $\mathcal{N}_\varepsilon(x_0) = \{x \mid |x - x_0| < \varepsilon\}$ and $\mathcal{N}^\circ_\varepsilon(x_0) = \{x \mid 0 < |x - x_0| < \varepsilon\}$ to denote the neighborhood of a point $x_0$. Then, the above remarks can be more precisely stated in the following proposition.

**Proposition 2.** *There exists an $\varepsilon > 0$ and a sufficiently small $\delta > 0$ such that for all $s \in \mathcal{N}_\delta(s_0)$, we may define $p_1(s)$ and $p_2(s)$ as the unique solution of (1) with $(p_1(s), p_2(s)) \in \mathcal{N}_\varepsilon(p_{10}, p_{20})$. The functions so defined are differentiable up to the third order.*

Note that, in general, for $s \in \mathcal{N}_\delta(s_0)$, characteristic equation (1) may have other solutions outside of $\mathcal{N}_\varepsilon(p_{10}, p_{20})$. We recall the stability crossing curves defined in Gu et al. (2005) as the set of all points $(p_1, p_2) \in \mathbb{R}^2_+$ such that $q(s)$ has at least one zero on the imaginary axis. Therefore, the set

$$\mathcal{T}_{(\omega_0, p_{10}, p_{20})} = \{(p_1(i\omega), p_2(i\omega)) \in \mathcal{N}_\varepsilon(p_{10}, p_{20}) \mid i\omega \in \mathcal{N}_\delta(i\omega_0)\},$$

which is a curve in the $p_1$-$p_2$ space that passes through the point $(p_{10}, p_{20})$, is the restriction of stability crossing curves to a neighborhood of $(p_{10}, p_{20})$. Thus, $\mathcal{T}_{(s_0, p_{10}, p_{20})}$ will be known as the *local stability crossing curve*. Roughly speaking, it is a curve that divides the neighborhood $\mathcal{N}_\varepsilon(p_{01}, p_{02})$ of the parameter space into regions, such that the number of characteristic roots on the right half complex plane remains constant as the parameters vary within each such region. We also define the positive and negative local stability crossing curves, corresponding to $\omega > \omega_0$ and $\omega < \omega_0$, respectively. For instance, we use the notation $\mathcal{T}^+_{(\omega_0, p_{10}, p_{20})} = \{(p_1(i\omega), p_2(i\omega)) \in \mathcal{N}_\varepsilon(p_{10}, p_{20}) \mid i\omega \in \mathcal{N}_\delta(i\omega_0), \omega > \omega_0\}$ for the positive local stability crossing curve. We point out that the stability crossing curves are known as the D-decomposition curves in Gryazina and Polyak (2006).

The purpose of this paper is to study how the two characteristic roots migrate as $(p_1, p_2)$ varies in a small neighborhood of $(p_{10}, p_{20})$ under the least degeneracy assumptions.

### 2.1. Cusp and local bijection

We parametrize a neighborhood of $s_0$ in the complex plane by using a radial variable $u$ and an angle $\theta$: $s = s_0 + ue^{i\theta}$. We also denote $\gamma = e^{i\theta} = \frac{\partial s}{\partial u}$. We can now fix the angular variable $\theta$, i.e., fix $\gamma$, and calculate the derivatives of $p_1$ and $p_2$ with respect to the radial variable $u$. This can be easily achieved by differentiating (1), yielding

$$\frac{\partial q}{\partial p_1}\frac{\partial p_1}{\partial u} + \frac{\partial q}{\partial p_2}\frac{\partial p_2}{\partial u} + \frac{\partial q}{\partial s}\gamma = 0. \tag{6}$$

If we set $u = 0$ and use the second equation of (3) in (6), we obtain

$$\begin{pmatrix} \mathrm{Re}\left(\dfrac{\partial q}{\partial p_1}\right) & \mathrm{Re}\left(\dfrac{\partial q}{\partial p_2}\right) \\ \mathrm{Im}\left(\dfrac{\partial q}{\partial p_1}\right) & \mathrm{Im}\left(\dfrac{\partial q}{\partial p_2}\right) \end{pmatrix}_{\substack{s=s_0\\ p_1=p_{10}\\ p_2=p_{20}}} \begin{pmatrix} \dfrac{\partial p_1}{\partial u} \\ \dfrac{\partial p_2}{\partial u} \end{pmatrix}_{u=0} = 0,$$