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Brief paper

A new frequency weighted Fourier-based method for model order reduction☆

Marek Rydel, Rafał Stanisławski *

Department of Electrical, Control and Computer Engineering, Opole University of Technology, ul. Prószkowska 76, 45-758 Opole, Poland

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ABSTRACT

This paper presents a new, analytically driven frequency weighted model order reduction method. The method is grounded on the Fourier-based decomposition of a high-order state space model. The method is designed for discrete-time systems, but it can be easily applied to continuous-time ones. The main advantage of the proposed algorithm is a class of quadratic time complexity as compared to the cubic one for the classical frequency weighted method, the major feature resulting from the application of analytical methods for calculation of factorizations for controllability and observability Gramians. The simulation experiment confirms the effectiveness of the proposed method both in terms of high modeling accuracy and low computational cost.

1. Introduction

Model order reduction is an effective method which has been applied in various theoretical and practical developments involving controls, circuit design, complex system modeling and analysis, approximation of fractional-order systems, and many others. The goal of the reduction is to generate a low-order model, which accurately describes complex system dynamics in a given adequacy range. It is worth mentioning that the reduction of high order systems is a time consuming process, in general. In this paper, we consider two model order reduction methods, that is Fourier model reduction (FMR) and frequency weighted (FW) method.

The FMR method (Willcox & Megretski, 2005) uses discrete-time Fourier coefficients to develop a model with a reduced number of state variables. Fourier coefficients can be efficiently calculated in an analytical way, which enables to obtain a very accurate approximation to the system dynamics. Usually, the obtained model contains only a fraction of state variables as compared to the original system. Furthermore, the obtained model, usually called the intermediate one, can be reduced for the second time by another model order reduction technique, e.g. balanced truncation approximation (BTA) method, see e.g. Antoulas (2005) and Moore (1981). As a result we have obtained the combined FMR–BTA method (Stanisławski, Rydel, & Latawiec, 2017). The BTA method is based on the Gramians' factorizations in order to determine transformation matrices which reduce the model. The method minimizes an approximation error at the whole frequency range, which sometimes is not necessary in case of a given adequacy range of the model. In that case it is more desirable to reduce model errors in a certain frequency interval rather than in the whole frequency range (Rydel & Stanisławski, 2015). The FW methods are an extended versions of BTA and they are based on a direct application of input/output weighting functions in the frequency domain. This leads to a better model accuracy in selected frequency ranges. The first FW method has been proposed by Enns (1984). However, despite of the simplicity of the method and successful employment in many applications, this method cannot guarantee the stability of the reduced model in case of two-sided weighting. Several modifications to the Enns method have been offered in the literature to cope with the stability problem (Campbell, Sreeram, & Wang, 2000; Ghafoor & Sreeram, 2007; Imran, Ghafoor, & Sreeram, 2014a; Lin & Chiu, 1992; Sahlan & Sreeram, 2009; Sreeram, Anderson, & Madievski, 1995; Sreeram & Sahlan, 2012; Varga & Anderson, 2003; Wan Muda, Sreeram, & Iu, 2011; Wang, Sreeram, & Liu, 1999). All those methods provide comparable frequency response errors and yield computable *a priori* error bounds. Proper selections of weighting functions enable a significant reduction of approximation errors for a given frequency range (Rydel & Stanisławski, 2015). Properties of weighting

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* Corresponding author.
E-mail addresses: m.rydel@po.opole.pl (M. Rydel), r.stanislawski@po.opole.pl (R. Stanisławski).

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functions are determined on the basis of frequency response of the model and a range of model adequacy. However, all those methods are of time complexity $\mathcal{O}(n^3)$.

In this paper we present a new model order reduction method, which is a combination of the FMR and FW methods and is called the FMR–FW method. The introduced algorithm is realized in two steps (1) the FMR method is applied to the original system in order to obtain the intermediate model and (2) the FW method is applied to the intermediate model generating the final reduced order model. The main advantage of the method are simple *analytical* formulae for selection of the controllability and observability Gramians' factorizations, which leads to reduction of computational complexity of the reduction algorithm.

This paper is organized as follows. Fundamentals of the FMR and FW methods are recalled in Section 2. The main result in terms of simple, analytical formulae for determination of the controllability and observability Gramians' factorizations is presented in Section 3. Consequently, this Section also contains a time complexity analysis for the introduced algorithm and presents some simplifications of the introduced algorithm both for the one-sided weighting and for SISO system cases. A numerical example of Section 4 confirms the effectiveness of the introduced methodology both in terms of modeling accuracy and low time complexity. Conclusions of Section 5 complete the paper.

2. Preliminaries

Consider a discrete-time LTI MIMO state-space system $G = \{A, B, C, D\}$ as follows

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ y(k) &= Cx(k) + Du(k) \qquad k = 0, 1, \ldots \end{aligned} \tag{1}$$

where $x(k) \in \Re^{n}$ is the state vector, $u(k) \in \Re^{n_u}$ and $y(k) \in \Re^{n_y}$ are the input and output vectors, respectively, $A \in \Re^{n\times n}$, $B \in \Re^{n\times n_u}$, $C \in \Re^{n_y \times n}$ and $D \in \Re^{n_y\times n_u}$.

2.1. Fourier method

It is well known that the system of Eq. (1) can be described in the transfer function matrix form

$$G(z) = C(Iz - A)^{-1}B + D \tag{2}$$

or using the Fourier decomposition

$$G(z) = \sum_{k=0}^{\infty} g_k z^{-k} \tag{3}$$

with $g_k \in \Re^{n_y\times n_u}$, $k = 0, 1, \ldots$, being the impulse response matrix components, calculated as

$$g_k = \begin{cases} D & k = 0 \\ CA^{k-1}B & k = 1, 2, \ldots. \end{cases} \tag{4}$$

Alternatively, g_k can be calculated in a recursive way

$$g_k = CH_{k-1},\ H_0 = B,\ H_k = AH_{k-1},\ k = 1, 2, \ldots. \tag{5}$$

A finite-length approximation of Eq. (3) can be obtained by bounding the summation process

$$\hat{G}(z) = \sum_{k=0}^{m} g_k z^{-k} \tag{6}$$

where g_k, $k = 0, \ldots, m$, are as in Eq. (4) or (5). An approximation error for the model (6) is analyzed in Willcox and Megretski (2005). The model based on Fourier series expansion (FMR) can be presented in the state space form $\hat{G} = \{\hat{A}, \hat{B}, \hat{C}, \hat{D}\}$ as

$$\begin{aligned} \hat{x}(k+1) &= \hat{A}\hat{x}(k) + \hat{B}u(k) \\ \hat{y}(k) &= \hat{C}\hat{x}(k) + \hat{D}u(k) \end{aligned} \tag{7}$$

where $\hat{x}(k) \in \Re^{n_f}$, $\hat{A} \in \Re^{n_f\times n_f}$, $\hat{B} \in \Re^{n_f\times n_u}$, $\hat{C} \in \Re^{n_y\times n_f}$, $\hat{D} \in \Re^{n_y\times n_u}$, $n_f = mn_u$, and the underlying matrices are as follows

$$\left[\begin{array}{c|c} \hat{A} & \hat{B} \\ \hline \hat{C} & \hat{D} \end{array}\right] = \left[\begin{array}{cccccc|c} 0 & 0 & 0 & \ldots & 0 & 0 & I \\ I & 0 & 0 & \ldots & 0 & 0 & 0 \\ 0 & I & 0 & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & I & 0 & 0 \\ \hline g_1 & g_2 & g_3 & \ldots & g_{m-1} & g_m & g_0 \end{array}\right] \tag{8}$$

with $I \in \Re^{n_u\times n_u}$ and $0 \in \Re^{n_u\times n_u}$ being the identity and zero matrices, respectively, and the components $g_k \in \Re^{n_y\times n_u}$, $k = 0, \ldots, m$, calculated as in Eq. (4) or (5).

It is well known that the number m of the Fourier elements can seriously affect the approximation error of the intermediate model and it has to be sufficiently large in order to produce a model of satisfactorily low approximation error. The outlined FMR method generates an intermediate model of order $n_f = mn_u$, which will be used in the next part of the paper.

2.2. Frequency weighted method

The FW method belongs to the class of the SVD-based methods and relies on the concept of balanced model realization (Antoulas, 2005; Moore, 1981), with application of weighting functions. The aim of the reduction algorithm is determination of a dominant part of the model, which can be obtained through calculation of the balancing (or transformation) matrix T and its inverse. The transformation matrix is not unique and there exist several algorithms to solve the inverse problem (Antoulas, 2005; Varga, 1991; Varga & Anderson, 2003). However, almost all of them are based on determination of the Cholesky factorizations of the controllability and observability Gramians, which is the first, most time consuming step in the whole algorithm. All subsequent steps of the FW algorithm are identical to those for the BTA method (Antoulas, 2005; Moore, 1981).

In the paper, we apply the FW method to the intermediate system given by Eq. (8), which can be transformed and partitioned as follows

$$T\hat{A}T^{-1} = \begin{bmatrix} A_r & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, T\hat{B} = \begin{bmatrix} B_r \\ B_2 \end{bmatrix}, \hat{C}T^{-1} = \begin{bmatrix} C_r & C_2 \end{bmatrix}$$

where the submatrices A_r, B_r, C_r describe the reduced-order state-space system $G_r = \{A_r, B_r, C_r, \hat{D}\}$ of order n_r, such that $n_r < n_f \ll n$.

The FW method has been developed for stable models with stable input and output weighting functions with minimal realizations $W_i = \{A_i, B_i, C_i, D_i\}$ and $W_o = \{A_o, B_o, C_o, D_o\}$ of orders n_i and n_o, respectively. Accounting that no pole-zero cancellations occur during forming of $\hat{G}W_i$ and $W_o\hat{G}$, the augmented systems are given as follows (Antoulas, 2005; Imran et al., 2014a; Wang et al., 1999)

$$\hat{G}W_i = \left[\begin{array}{c|c} \tilde{A}_i & \tilde{B}_i \\ \hline \tilde{C}_i & \tilde{D}_i \end{array}\right] = \left[\begin{array}{cc|c} \hat{A} & \hat{B}C_i & \hat{B}D_i \\ 0 & A_i & B_i \\ \hline \hat{C} & \hat{D}C_i & \hat{D}D_i \end{array}\right]$$

$$W_o\hat{G} = \left[\begin{array}{c|c} \tilde{A}_o & \tilde{B}_o \\ \hline \tilde{C}_o & \tilde{D}_o \end{array}\right] = \left[\begin{array}{cc|c} \hat{A} & 0 & \hat{B} \\ B_o\hat{C} & A_o & B_o\hat{D} \\ \hline D_o\hat{C} & C_o & D_o\hat{D} \end{array}\right].$$