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Brief paper

Stochastic output feedback control: Convex lifting approach☆

Ngoc Anh Nguyen

Laboratory of Signals and Systems, CentraleSupélec-CNRS-UPS, Université Paris Saclay, Gif-sur-Yvette, France

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ABSTRACT

This paper presents a method to design output feedback control for discrete-time linear systems, affected by bounded additive state, output disturbances, and subject to chance constraints on the state and hard constraints on the control input. This method relies on a so-called convex lifting which is a nonnegative, convex, piecewise affine function, equal to 0 over a given stochastic positively invariant set and strictly positive outside this set. Accordingly, it is shown that this function is strictly decreasing along the closed-loop dynamics outside this invariant set and convergent to 0 as time tends to infinity. Consequently, the state is convergent to the given invariant set, while the method only requires solving a linear program at each sampling instant.

1. Introduction

Stochastic control has recently received growing interest, as it provides a relaxation over robust control. This relaxation is represented by the allowance of an acceptable rate of failures, since the worst cases are usually unlikely. In the language of stochastic programming, e.g. in Prékopa (2013), these failures are modeled by constraints' violation, i.e., the probability of constraints' violation is smaller than a pre-defined rate. Such constraints are usually referred to as *chance/probabilistic* constraints. Control design for systems subject to those constraints needs new development in comparison to its counterpart subject to deterministic constraints, a survey on this topic is presented in Calafiore, Dabbene, and Tempo (2011). In Cannon, Kouvaritakis, Raković, and Cheng (2011) and Kouvaritakis, Cannon, Raković, and Cheng (2010), the authors present approaches relying on model predictive control (MPC), where closed-loop stability is guaranteed by means of suitable terminal constraints. However, these approaches cannot guarantee the recursive feasibility if the closed loop goes beyond the feasible region. This limitation is resolved in Lorenzen, Allgower, Dabbene, and Tempo (2015) by adding a constraint ensuring that the state stays inside the feasible region despite any disturbances in the given bounded set. Accordingly, the problem reduces to solving a quadratic program at each sampling instant. On the other hand, to deal with this control design problem, Calafiore and Fagiano (2013) present a scenario approach in the context of MPC. The method does not require the convexity of the disturbance sets, its online computation becomes however much more expensive than the aforementioned methods, since a sufficiently large number of samples are chosen to guarantee the chance constraints, leading to an exponential number of constraints along with the prediction horizon. Recall that the boundedness of disturbances is not assumed therein, but it is implicitly accounted for since the number of samples is finite. In addition to the above MPC approaches, set-theoretic approach is also proposed in Kofman, De Doná, and Seron (2012). This study introduces a computation technique of a *probabilistic invariant set*, which guarantees the positive invariance with a big enough probability, while not assuming the boundedness of disturbances. However, constraints are not accounted for therein. Furthermore, making use of this technique to compute the feasible region is not trivial.

When the measurement noise is considered, the problem becomes output feedback control design subject to chance constraints. Control design in this case is more difficult, as a suitable observer is designed to estimate the state, which leads to the fact that the error between the real state and the estimated state is not independent, identically distributed disturbance. Although many studies have been dedicated to stochastic output feedback control design, e.g. Boukas (2006) and Deng and Krstic (1999), most of them do not take constraints into account which is a crucial problem, since it is directly related to the determination of the feasible region.

Note also that set-theoretic methods are shown in Blanchini (1994) and Nguyen, Olaru, Rodríguez-Ayerbe, and Kvasnica (2017) to be cheaper than MPC in the robust context. Motivated by the benefit of set-theoretic methods, this paper presents the so-called convex lifting approach in the context of stochastic control, which does not require terminal constraints to ensure closed-loop stability as in MPC methods. Moreover, chance constraints on the

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E-mail address: Ngocanh.Nguyen.rs@gmail.com.

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state and hard constraints on the control variable are taken into account, allowing for a feasible region larger than the one in the robust case. To this end, computation of a stochastic positively invariant set and of the feasible region is proposed to cope with the given chance constraints by means of relevant convex inner approximations which only depend on the mean value and the standard deviation of the disturbances. Unlike the feasible region deployed in Blanchini (1994) and Nguyen, Kvasnica et al. (2017), the one presented in this paper does not inherit the contractivity property. Subsequently, we construct a real-valued, nonnegative, convex, piecewise affine function defined over the feasible region, usually referred to as a convex lifting. In particular, unlike a control Lyapunov function which is equal to 0 only at the origin, this function is instead equal to 0 over the given stochastic positively invariant set and strictly positive outside this set. Accordingly, it is shown that this function is strictly decreasing along the closed-loop dynamics outside this set and convergent to 0, as time tends to infinity, leading to the fact x_k converges to the given stochastic positively invariant set. Moreover, the proposed control design only requires solving a linear program at each sampling instant. Therefore, this method might be useful for systems with fast dynamics like cantilever beam system. More clearly, either implicit or explicit solution could be deployed at the hardware level, the real-time implementation of explicit solution for this system based on convex lifting is referred to Gulan et al. (2017a, 2017b).

Nomenclature: throughout this paper, $\mathbb{N}$, $\mathbb{N}_{>0}$, $\mathbb{R}$, $\mathbb{R}_+$ denote the set of nonnegative integers, the set of positive integers, the set of real numbers and the set of nonnegative real numbers, respectively. Also, I denotes an identity matrix of suitable dimension. For ease of presentation, we denote the index set $\mathcal{I}_N := \{1, 2, \dots, N\}$ with $N \in \mathbb{N}_{>0}$. A polyhedron is the intersection of finitely many closed halfspaces. A polytope is a bounded polyhedron. If P is an arbitrary polytope, then $\mathcal{V}(P)$ denotes the set of its vertices. If $\mathcal{S}$ is an arbitrary set, then $\operatorname{conv}(\mathcal{S})$ denotes the convex hull of $\mathcal{S}$. Given a set $\mathcal{S} \subset \mathbb{R}^d$ and a matrix $A \in \mathbb{R}^{m\times d}$, then $A\mathcal{S} := \{As : s \in \mathcal{S}\}$. Also, for a vector $x \in \mathbb{R}^d$, define $\rho_{\mathcal{S}}(x) := \min\limits_{y\in\mathcal{S}} \sqrt{(y-x)^T(y-x)}$. Given two sets $\mathcal{S}_1, \mathcal{S}_2 \subset \mathbb{R}^d$, their Minkowski sum $\mathcal{S}_1 \oplus \mathcal{S}_2$ is defined as: $\mathcal{S}_1 \oplus \mathcal{S}_2 := \left\{y_1 + y_2 \in \mathbb{R}^d : y_1 \in \mathcal{S}_1, y_2 \in \mathcal{S}_2\right\}$. Also, $\mathcal{S}_1 \setminus \mathcal{S}_2$ is defined as follows: $\mathcal{S}_1 \setminus \mathcal{S}_2 := \left\{x \in \mathbb{R}^d : x \in \mathcal{S}_1,\ x \notin \mathcal{S}_2\right\}$. Given a random variable $\xi \in \mathbb{R}^d$, we use $\mathrm{E}(\xi)$, $\operatorname{cov}(\xi)$ to represent the mean value and the covariance matrix of ξ. $\Pr(\cdot)$ implies the probability of an event.

2. Problem settings

In this paper, we consider a discrete-time linear system:

$$x_{k+1} = Ax_k + Bu_k + w_k,\ \ y_k = Cx_k + v_k, \tag{1}$$

where x_k, u_k denote the state, control variable at time k and w_k, v_k represent the additive state and output disturbances at time k, respectively. We assume that w_k, v_k for all $k \in \mathbb{N}$ are zero-mean, mutually independent random variables and they fulfill:

$$w_k \in \mathbb{W},\ v_k \in \mathbb{V}.$$

Also, suppose system (1) satisfies the following properties.

Assumption 2.1. The pair $(A,\ B)$ is controllable and the pair $(A,\ C)$ is observable.

As the state is allowed for an acceptable rate of constraints' violation, it is subject to the following chance constraints:

$$\Pr(x_k \in \mathbb{X}) \geq 1 - \alpha, \tag{2}$$

where $\alpha \in (0, 1)$ represents a given constant scalar. On the other hand, the control variable is subject to hard constraints:

$$u_k \in \mathbb{U}. \tag{3}$$

With respect to given $d_x, d_u, d_y \in \mathbb{N}_{>0}$, it is assumed that the sets $\mathbb{X} \subset \mathbb{R}^{d_x}$, $\mathbb{U} \subset \mathbb{R}^{d_u}$, $\mathbb{W} \subset \mathbb{R}^{d_x}$, $\mathbb{V} \subset \mathbb{R}^{d_y}$ are polytopes containing the origin in their interior.

This paper aims to present a control design method which can both stabilize system (1) and satisfy the chance constraint (2) and the hard constraint (3). A common approach to dealing with measurement noise is to make use of a Luenberger observer, see further in Luenberger (1964):

$$\hat{x}_{k+1} = A\hat{x}_k + Bu_k + L(y_k - \hat{y}_k),\ \ \hat{y}_k = C\hat{x}_k. \tag{4}$$

Accordingly, if one defines $e_k = x_k - \hat{x}_k$, then the behavior of e_k is described by the following autonomous system:

$$e_{k+1} = (A - LC)e_k + w_k - Lv_k. \tag{5}$$

Since $w_k \in \mathbb{W}$, $v_k \in \mathbb{V}$, the disturbance of system (5) satisfies $w_k - Lv_k \in \mathbb{W} \oplus (-L\mathbb{V})$. If one chooses a suitable observer gain L such that $A - LC$ is strictly stable, then there exists a robust positively invariant set denoted by Ω_e for system (5) according to Gilbert and Tan (1991). Note also that the observer system (4) can be written in the form:

$$\hat{x}_{k+1} = A\hat{x}_k + Bu_k + Lv_k + LCe_k,\ \ \hat{y}_k = C\hat{x}_k. \tag{6}$$

Accordingly, if one considers $Lv_k + LCe_k$ as additive disturbance of system (6) and $e_0 \in \Omega_e$, then the robust positive invariance of Ω_e yields $Lv_k + LCe_k \in L\mathbb{V} \oplus LC\Omega_e$. As a consequence, the control design problem for system (1) subject to the constraints (2), (3) is translated to the one for system (6) subject to the following:

$$\Pr(\hat{x}_k + e_k \in \mathbb{X}) \geq 1 - \alpha, \tag{7a}$$

$$u_k \in \mathbb{U},\ Lv_k + LCe_k \in L\mathbb{V} \oplus LC\Omega_e, \tag{7b}$$

where e_k for all $k \in \mathbb{N}$ are not mutually independent, but follow the dynamic equation (5) and their mean value depends on time and the initial condition. For ease of presentation, let Σ_0, Σ_w, Σ_v denote the covariance matrices of e_0, w_k and v_k, respectively. Assume that the initial condition e_0 fulfills:

Assumption 2.2. e_0 is a zero-mean random variable, independent with w_k, v_k for all $k \in \mathbb{N}$. Also, $e_0 \in \Omega_e$ and its covariance matrix satisfies:

$$\Sigma_0 \leq \sum_{j=0}^{\infty} (A - LC)^j (\Sigma_w + L\Sigma_v L^T) \left((A - LC)^j\right)^T.$$

One can easily see that $\mathrm{E}(e_{k+1}) = 0$ and the covariance matrix of e_{k+1} is determined as follows according to (5):

$$\operatorname{cov}(e_{k+1}) = (A - LC)^{k+1}\Sigma_0((A - LC)^{k+1})^T + \sum_{i=0}^{k} (A - LC)^i (\Sigma_w + L\Sigma_v L^T)((A - LC)^i)^T.$$

According to Assumption 2.2, the above equation yields $\Sigma_e \geq \operatorname{cov}(e_k)$ for all $k \in \mathbb{N}$, where

$$\Sigma_e = \sum_{j=0}^{\infty} (A - LC)^j (\Sigma_w + L\Sigma_v L^T) \left((A - LC)^j\right)^T.$$

Note also that $A - LC$ is strictly stable, Σ_e is thus upper bounded and can be computed by the Lyapunov equation

$$\Sigma_e = \Sigma_w + L\Sigma_v L^T + (A - LC)\Sigma_e(A - LC)^T. \tag{8}$$