Automatica ■ (■■■■) ■■■–■■■

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Technical communique

A simple necessary and sufficient LMI condition for the strong delay-independent stability of LTI systems with single delay[☆]

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ARTICLE INFO

Article history:
Received 3 July 2017
Received in revised form 26 September 2017
Accepted 26 September 2017
Available online xxxx

Keywords:
Time-delay
Delay-independent stability
Kalman–Yakubovich–Popov lemma
Kronecker sum properties
LMIs

ABSTRACT

This paper is concerned with strong delay-independent stability of linear time-invariant (LTI) systems with a single time-delay. Stability analysis of linear delay-systems is complicated by the need to locate the roots of a transcendental characteristic equation. In this paper we propose a convex necessary and sufficient condition for strong delay-independent stability. This result mainly follows from the Kronecker sum properties and the Kalman–Yakubovich–Popov lemma, which allows us to present the main result in terms of a single linear matrix inequality (LMI) feasibility test. The result is illustrated by simple numerical examples.

1. Introduction

Existing methods for the stability analysis of time-delay systems in the literature often fall into two categories: delay-independent stability and delay-dependent stability, depending on whether or not stability has to be maintained for all positive delay values. Recent results and surveys of the literature can be found in Briat (2015), Fridman (2014), Gu, Kharitonov, and Chen (2003), Kharitonov (1999) and Niculescu (2001).

The present paper focuses on delay-independent stability. The goal is to characterize whether the origin of a linear state-delayed system of the form

$$\dot{x}(t) = Ax(t) + Bx(t-\tau), \tag{1}$$

where $x(t) \in \mathbb{R}^n$, $A, B \in \mathbb{R}^{n\times n}$, is an asymptotically stable equilibrium point for all possible values of the delay parameter $\tau \geq 0$, and the initial condition is $x(t) = \varphi(t)$, $\varphi : [-\tau,\ 0] \to \mathbb{R}^n$. It is known that for a given delay, τ, the delayed system in (1) is asymptotically stable if and only if all roots of the transcendental function

$$\Delta(s, e^{-s\tau}) := \det(sI - A - Be^{-s\tau}) \tag{2}$$

lie in the open left half of the complex plane.

One can generally find in the literature two notions of delay-independent stability for linear delay-systems, both related to the following polynomial in two variables:

$$\Delta(s, z) := \det(sI - A - Bz). \tag{3}$$

The first and less restrictive is the delay-independent stability characterization:

Definition 1. System (1) is delay-independent stable if $\Delta(s, z) \neq 0$ with $z = e^{-s\tau}$, $\forall s \in \mathbb{C}_+$, and $\tau \geq 0$.

In this definition, $\mathbb{C}_+$ denotes the closed right half plane of the complex plane. A slightly more restrictive condition is the *strong* notion of delay-independent stability.

Definition 2. System (1) is strongly delay-independent stable if $\Delta(s, z) \neq 0$, $\forall s \in \mathbb{C}_+$, and $z \in \mathbb{D}$.

The symbol $\mathbb{D}$ denotes the closed unit disc. Comparing the above definitions, *strong* delay-independent stability is defined by regarding s and $e^{-\tau s}$ as completely independent variables. For this reason, *strong* delay-independent stability is slightly stricter than the former. In fact, the only difference is at the origin, i.e. $s = 0$,

☆ This work has been supported in part by the Brazilian agencies CNPq, FAPEMIG, and CAPES. The material in this paper was not presented at any conference. This paper was recommended for publication in revised form by Associate Editor Jamal Daafouz under the direction of Editor André L. Tits.

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https://doi.org/10.1016/j.automatica.2017.11.006
0005-1098/

where $z = e^{-\tau s} = 1 \subset \mathbb{D}$. On the other hand, *strong* delay-independent stability is often more robust against perturbations in the matrices A and B (Bliman, 2002).

Delay-independent stability has been studied in the literature using a myriad of techniques. The interested reader is referred to Bliman (2002), Chen and Latchman (1995), Fridman (2014), Gu et al. (2003), He, Wang, Lin, and Wu (2005), Kharitonov (1999), Li, Gao, and Gu (2016), Niculescu (2001) and Souza, de Oliveira, and Palhares (2009) for more details. Most delay-independent stability conditions in the literature are only sufficient. That is, they may be able to verify stability for certain systems but fail for others (see, for instance, Gu et al. (2003, Section 3.5)). Their main advantage is the generally lower computational cost. Many of these sufficient conditions can be formulated in terms of linear matrix inequalities (LMIs) and, from there, be generalized to cope with problems of filters and controller design (de Oliveira & Geromel, 2004; de Souza, Palhares, & Dias Peres, 2001).

Necessary and sufficient delay-independent conditions exist but often involve difficult computations. In Bliman (2002), a method was proposed for verifying *strong* delay-independent stability based on a family of LMIs of increasing dimensions. The LMI conditions approach necessity for a large enough dimension. A well known necessary and sufficient delay-independent condition in the case of a single delay is the frequency-sweeping test:

Lemma 1 (*Chen and Latchman, 1995, Theorem 3.1*). *The time-delay system in (1) is delay-independent stable if and only if*

(i) *A is Hurwitz;*
(ii) $\rho\big((j\omega I - A)^{-1}B\big) < 1, \ \forall\, \omega > 0$;
(iii) *either*

  (a) $\rho(A^{-1}B) < 1$ *or*
  (b) $\rho(A^{-1}B) = 1$ *and* $\det(A + B) \neq 0$.

Furthermore, if (ii) *also holds for* $\omega = 0$ *then the system in (1) is strongly delay-independent stable.*

In the above lemma, $\rho(X)$ denotes the spectral radius of the square matrix X. The infinite-dimensional condition in Lemma 1 cannot be directly translated into a finite dimensional convex optimization problem. The main obstacle is the evaluation of the spectral radius on a range of frequencies. It can be, however, evaluated graphically, by plotting $\rho\big((j\omega I - A)^{-1}B\big)$ on a fine grid. This approach may lack accuracy in certain cases and also does not generalize to problems of filter and controller design. See Chesi and Middleton (2014) for some recent results in this direction and Chen and Latchman (1995) for further discussions on the notions of delay-independent stability in the context of the frequency-sweeping test. Recently, in Li et al. (2016), a new technique was proposed that overcome the need for a fine grid in a *strong* version of the frequency-sweeping test by discretizing the frequency domain into several sub-intervals and employing a piecewise constant Lyapunov matrix to analyze the frequency-dependent stability condition. This approach leads to an exact LMI characterization of *strong* delay-independent stability as the number of frequency points is increased. The main drawback is that the number of required LMI tests increases with the number of frequency points. The method also requires an iterative procedure for refining the domains.

In the present paper we show how to construct a necessary and sufficient *strong* delay-independent stability condition based on a single LMI test. In order to construct the main result we first employ a property of the Kronecker sum in order to translate the stability analysis problem into a problem of detecting a singularity in a certain frequency-dependent matrix. Then, we show how to replace the frequency dependence on the resulting condition by a matrix valued decision variable. The main technical result used then is the well-known Kalman–Yakubovich–Popov (KYP) lemma (Rantzer, 1996), which converts an infinite dimensional frequency domain inequality into a finite dimensional LMI. The KYP lemma is stated next:

Lemma 2 (*KYP Lemma*). *Given matrices $A_\#$, $B_\#$, and Q of compatible dimensions, the infinite dimensional frequency domain inequality*

$$\begin{bmatrix}(e^{j\theta}I - A_\#)^{-1}B_\# \\ I\end{bmatrix}^* Q \begin{bmatrix}(e^{j\theta}I - A_\#)^{-1}B_\# \\ I\end{bmatrix} \succ 0 \tag{4}$$

holds for all $\theta \in \mathbb{R}$ if and only if

$$\begin{bmatrix}A_\# & B_\# \\ I & 0\end{bmatrix}^T \begin{bmatrix}-P & 0 \\ 0 & P\end{bmatrix} \begin{bmatrix}A_\# & B_\# \\ I & 0\end{bmatrix} + Q \succ 0 \tag{5}$$

holds for some symmetric matrix P.

The resulting LMI, which will be presented in Theorem 1 in Section 2, can be efficiently solved numerically using polynomial-time algorithms for convex optimization with constraints defined by LMIs (Gahinet, Nemirovski, Laub, & Chilali, 1995; Toh, Todd, & Tütüncü, 1999).

The main advantage of the method proposed in the present paper as compared, for instance with those from Bliman (2002) and Li et al. (2016), is the need to test only a single finite dimensional LMI. The applicability and effectiveness of the main result will be illustrated by examples in Section 3.

Notation throughout the paper is standard. The symbol M^T (M^*) denotes the transpose (conjugate) of matrix M. If X is square and Hermitian then $X \succ 0$ ($X \prec 0$) indicates that X is positive (negative) definite. $\rho(X)$ denotes the spectral radius of the square matrix X. The notations $\otimes$ and $\oplus$ denote the Kronecker product and Kronecker sum, respectively.

2. The main results

We start with the following technical result.

Lemma 3. *Let $F(\theta) = A + Be^{-j\theta}$. The following statements are equivalent:*

(i) *System (1) is strongly delay-independent stable;*
(ii) *Matrix $F(\theta)$ is Hurwitz for all $\theta \in [0,\ 2\pi]$;*
(iii) *Matrix $F(0) = A + B$ is Hurwitz and*

$$\det\big(F(\theta) \oplus F^*(\theta)\big) \neq 0$$

for all $\theta \in [0,\ 2\pi]$.

Proof. The proof of equivalence between items (*i*) and (*ii*) can be found in Kamen (1982), see also Li et al. (2016, Lemma 1) and references therein. The equivalence between (*ii*) ⇔ (*iii*) is shown next.
(*ii*) ⇒ (*iii*): First recall that the eigenvalues of the Kronecker sum of the matrices $F(\theta)$ and $F^*(\theta)$ are all possible pairwise sums of the eigenvalues of $F(\theta)$ and $F^*(\theta)$ (see e.g. Niculescu (2001, Proposition C.2), Bernstein (2009, Proposition 7.2.3), and Horn & Johnson (1994, Theorem 4.4.5)). Therefore, if $F(\theta)$ is Hurwitz, then $F(\theta) \oplus F^*(\theta)$ is also Hurwitz and $\det\big(F(\theta) \oplus F^*(\theta)\big) \neq 0$.
(*ii*) ⇐ (*iii*): If (*iii*) holds, then $A + B$ is Hurwitz, that is $(A + Be^{-j\theta})$ is Hurwitz for $\theta = 0$. Because of the continuity of the eigenvalues of $(A+Be^{-j\theta})$ with respect to θ and the property of the Kronecker sum previously discussed, the matrix $(A+Be^{-j\theta})$ remains Hurwitz for all $\theta \in [0,\ 2\pi]$. This is true since for the real part of an eigenvalue of $F(\theta)$ to become non-negative, an eigenvalue of $F(\theta) \oplus F^*(\theta)$ should first become zero. □

The following auxiliary result is needed to prove the main result.

Please cite this article in press as: Souza, F.O., et al., A simple necessary and sufficient LMI condition for the strong delay-independent stability of LTI systems with single delay. Automatica (2017), https://doi.org/10.1016/j.automatica.2017.11.006.