ARTICLE IN PRESS

Automatica ■ (■■■■) ■■■–■■■

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Technical communique

A Newton-like algorithm to compute the inverse of a nonlinear map that converges in finite time ☆

Laura Menini, Corrado Possieri *, Antonio Tornambè

Dipartimento di Ingegneria Civile e Ingegneria Informatica, Università di Roma Tor Vergata, Via del Politecnico 1, 00133 Roma, Italy

ARTICLE INFO

Article history:
Received 15 May 2017
Received in revised form 28 August 2017
Accepted 2 November 2017
Available online xxxx

ABSTRACT

This paper deals with the problem of inverting a nonlinear map. The proposed solution consists in a nonlinear state observer, which mimics a Newton-like algorithm, that allows to determine the inverse of a given diffeomorphism in finite time. The results are illustrated by application to the inverse kinematics of a three DOF planar manipulator.

1. Introduction

The problem of determining the inverse of a nonlinear map is a central issue in many fields as, *e.g.*, robotics and observer design. The inverse of the observability map can be used to design observers able to estimate the state of a nonlinear system (Gauthier & Kupka, 2001; Menini, Possieri, & Tornambe, 2016), whereas the inverse of the direct kinematics of a robotic manipulator can be used to define its position in terms of the joint coordinates (Hartenberg & Denavit, 1955). In some cases, the exact inverse of a nonlinear map can be computed (Sturmfels, 2002), but its closed-form may be cumbersome (Goldenberg, Benhabib, & Fenton, 1985).

In Nicosia, Tornambe, and Valigi (1991b), the problem of inverting a nonlinear map has been reduced to the construction of a nonlinear observer, whereas in Nicosia, Tornambe, and Valigi (1992) it has been shown that the Newton algorithm can be interpreted as a state observer. Recently, in Blanchini, Fenu, Giordano, and Pellegrino (2017), a method is given to steer to a desired value the output of an unknown function, whose Jacobian takes values in a known polytope. In this work, an approach similar to the one given in Nicosia et al. (1992) (*i.e.*, a state observer) is used to design a technique able to invert a nonlinear map. Differently from the approaches given in Blanchini et al. (2017), Nicosia, Tornambe, and Valigi (1991a), Nicosia et al. (1991b, 1992); Nicosia, Tornambe, and Valigi (1994), the proposed observer converges in finite time to the inverse, thus improving the performances of the previously existing procedures. The effectiveness of the method is illustrated by computing the inverse kinematics of a robotic manipulator.

2. A Newton-like algorithm for the inverse of a nonlinear map that converges in finite time

Let $\mathbb{R}$, $\mathbb{R}_{\geqslant 0}$, $\mathbb{R}_{>0}$, and $\mathbb{Z}$ denote the set of real, nonnegative real, positive real, and integer numbers. Let $\sigma(A)$ denote the spectrum of the square matrix A. Letting $v = [v_1 \ \cdots \ v_n]^\top \in \mathbb{R}^n$, $\|v\|_j := (\sum_{i=1}^{n}|v_i|^j)^{\frac{1}{j}}$ is the j-norm of v. Letting $\mathcal{A}$, $\mathcal{B}$ be suitable subsets of $\mathbb{R}^n$, let

$$h : \mathcal{A} \to \mathcal{B},$$

be a diffeomorphism, *i.e.*, a smooth, bijective mapping whose inverse is smooth (Appendix A.3 of Isidori, 2013). For any positive and sufficiently small $\varepsilon > 0$, define

$$\Omega_\varepsilon := \{\xi \in \mathbb{R}^n : \|\xi\|_2 < \varepsilon\}, \tag{1a}$$

$$\mathcal{A}_\varepsilon := \{\xi \in \mathcal{A} : \hat{\xi} := \xi - \tilde{\xi} \in \mathcal{A}, \ \forall \tilde{\xi} \in \Omega_\varepsilon\} \tag{1b}$$

and consider the following problem.

Problem 1. Let $\varepsilon > 0$ be given, let $\mathcal{A}_\varepsilon$ and Ω_ε be defined as in (1), let $\mathcal{A}_\varepsilon$ be nonempty, let the mapping $x : \mathbb{R}_{\geqslant 0} \to \mathcal{A}_\varepsilon$ be $\mathcal{C}^0$ and piecewise $\mathcal{C}^1$, and let $y(t) = h(x(t))$ for all times $t \in \mathbb{R}$, $t > 0$. Letting $\dot{y}(t) = \frac{d}{dt}y(t)$, find a mapping $f : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ such that each solution of

$$\dot{\hat{x}} = f(\hat{x}, y, \dot{y}), \tag{2}$$

with $\tilde{x}(0) := x(0) - \hat{x}(0) \in \Omega_\varepsilon$, is such that $\tilde{x}(t) := x(t) - \hat{x}(t)$ *converges to* 0 *in finite time*, *i.e.*, there exists $T \in \mathbb{R}$, $T \geqslant 0$, such that $x(t) - \hat{x}(t) = 0$, for all $t > T$.

Remark 1. The system (2) considered in Problem 1 takes as inputs the output of the map $h(x(t))$ and its time derivative. If the latter signal cannot be directly measured, it can be however

☆ The material in this paper was not presented at any conference. This paper was recommended for publication in revised form by Associate Editor A. Pedro Aguiar under the direction of Editor André L. Tits.

* Corresponding author.
E-mail addresses: menini@disp.uniroma2.it (L. Menini), possieri@ing.uniroma2.it (C. Possieri), tornambe@disp.uniroma2.it (A. Tornambè).

https://doi.org/10.1016/j.automatica.2017.12.021
0005-1098/

estimated exactly and in finite time through techniques taken from the literature, as, for instance, the one proposed in Levant (1998).

The solution to Problem 1 proposed in this work consists in a state observer (a modified version of the Newton algorithm) that allows to achieve finite-time convergence to zero of the estimation error. Namely, let

$$J(x) := \frac{\partial}{\partial x} h(x),$$

which has full rank for all $x \in \mathcal{A}$ (Appendix A.3 of Isidori, 2013). Hence, consider the system

$$\dot{\hat{x}} = J^{-1}(\hat{x})(\dot{y} + k\,\|\tilde{y}\|_2^{\alpha}\,\mathrm{sign}(\tilde{y})), \tag{3}$$

where $k > 0, \alpha > 0, \alpha < 1, \tilde{y} = y - h(\hat{x})$, and $\mathrm{sign}(\cdot)$ denotes the entry-wise sign operator. Due to the discontinuity of (3), in the following, solutions to system (3) are understood in the Filippov sense (Filippov, 1988).

The following theorem states that system (3) is a solution to Problem 1.

Theorem 1. *Let the assumptions of Problem 1 hold and let $\hat{x}(0)$ be such that $\tilde{x}(0) \in \Omega_\varepsilon$. Then, letting $\hat{x}(t)$ be the solution to system (3), there exists $T \in \mathbb{R}$, $T \geqslant 0$, such that $\tilde{x}(t) = x(t) - \hat{x}(t) = 0$ for all times $t \in \mathbb{R}$, $t > T$.*

Proof. Consider the function

$$V = \tfrac{1}{2}\tilde{y}^\top\tilde{y} = \tfrac{1}{2}\|\tilde{y}\|_2^2, \tag{4}$$

that is positive definite with respect to $\tilde{x} := x - \hat{x}$, provided that $x \in \mathcal{A}_\varepsilon$ and $\tilde{x} \in \Omega_\varepsilon$. In fact, define the function of the scalar θ, parametric with respect to $x \in \mathcal{A}_\varepsilon$ and $\tilde{x} \in \Omega_\varepsilon$, $w(\theta) := h(x - \theta\tilde{x})$. By Theorem 5.19 of Rudin (1964) applied to $w(\theta)$, that is a differentiable function of θ in [0, 1], there exists $\bar{\theta} \in [0, 1]$ such that

$$\|\tilde{y}\|_2 = \|w(1) - w(0)\|_2 \leqslant \|J(x - \bar{\theta}\tilde{x})\|_2\|\tilde{x}\|_2.$$

Hence, $\tilde{x} = 0$ implies $V = 0$. In order to prove the positive definiteness of V with respect to $\tilde{x}$, assume, by contradiction, that there exists $x \in \mathcal{A}_\varepsilon$ and $\tilde{x} \in \Omega_\varepsilon$, $\tilde{x} \neq 0$, such that $V = 0$. Since $V = 0$ if and only if $\tilde{y} = 0$, this implies that $h(x) = h(x - \tilde{x})$ for some $\tilde{x} \neq 0$, leading to a contradiction because, by assumption, h is a diffeomorphism from $\mathcal{A}$ to $\mathcal{B}$.

By computing the time derivatives of V along the trajectories of system (3), one obtains that

$$\begin{aligned}\dot{V} &= \tilde{y}^\top\dot{\tilde{y}} = \tilde{y}^\top\left(\dot{y} - \frac{\partial h(\hat{x})}{\partial x}\dot{\hat{x}}\right)\\ &= \tilde{y}^\top(\dot{y} - J(\hat{x})J^{-1}(\hat{x})(\dot{y} + k\,\|\tilde{y}\|_2^{\alpha}\,\mathrm{sign}(\tilde{y})))\\ &= -k\,\tilde{y}^\top\mathrm{sign}(\tilde{y})\|\tilde{y}\|_2^{\alpha} = -k\|\tilde{y}\|_1\|\tilde{y}\|_2^{\alpha},\end{aligned}$$

for all $x \in \mathcal{A}_\varepsilon$ and $\tilde{x} \in \Omega_\varepsilon$. Hence, since $\|\cdot\|_1 \geqslant \|\cdot\|_2$,

$$\dot{V} \leqslant -k\|\tilde{y}\|_2^{\alpha+1} = -k\,V^{\frac{\alpha+1}{2}}. \tag{5}$$

Therefore, if $\hat{x}(0)$ is such that $\tilde{x}(0) \in \Omega_\varepsilon$, by (5) and Bhat and Bernstein (2000), there exists a time $T \in \mathbb{R}$, $T > 0$, such that $V(t) = 0$ for all times $t \in \mathbb{R}$, $t \geqslant T$, and hence $\tilde{y}(t) = 0$ for all $t \geqslant T$. Thus, since V is positive definite with respect to $\tilde{x}$, one has that $\tilde{x}(t) = 0$ for all $t \geqslant T$. □

The inequality given in (5) provides an upper bound on the convergence of the function V to zero that depends on the design parameters k and α of (3). Namely, letting $T^\star = \frac{1}{k(1-\alpha)}2^{\frac{\alpha+1}{2}}\|\tilde{y}(0)\|_2^{1-\alpha}$, by (5), one has that

$$V(t) \leqslant \begin{cases}(\frac{1}{2}(\alpha-1)kt + 2^{\frac{\alpha-1}{2}}\|\tilde{y}(0)\|_2^{1-\alpha})^{\frac{2}{1-\alpha}}, & \text{if } t \leqslant T^\star,\\ 0, & \text{if } t > T^\star,\end{cases}$$

for all $t \in \mathbb{R}_{\geqslant 0}$. Thus, letting $\delta := \sup_{\tilde{x}\in\Omega_\varepsilon,\, x\in\mathcal{A}_\varepsilon}\|h(x) - h(x - \tilde{x})\|_2$ be the maximum admissible initial error in $\tilde{y}$ the finite convergence time T satisfies

$$T \leqslant \frac{2^{\frac{\alpha+1}{2}}\delta^{1-\alpha}}{k(1-\alpha)}. \tag{6}$$

Assume now that just estimates $\hat{J}$ and $\hat{\dot{y}}$ of J and $\dot{y}$ are known and consider the following system

$$\dot{\hat{x}} = \hat{J}^{-1}(\hat{x})(\hat{\dot{y}} + k\,\|\tilde{y}\|_2^{\alpha}\,\mathrm{sign}(\tilde{y})), \tag{7}$$

that is system (3) where J and $\dot{y}$ have been substituted with their estimates $\hat{J}$ and $\hat{\dot{y}}$, respectively. The following theorem proves the robustness of the proposed solution to Problem 1 with respect to such uncertainties.

Theorem 2. *Let the positions of Theorem 1 hold and, letting $G(x) := J(x)\hat{J}^{-1}(x)$, assume, additionally, that*

(i) $0 < \inf_{x\in\mathcal{A}}|\det(J(x))|$ *and* $\sup_{x\in\mathcal{A}}\|J(x)\|_2 < \infty$;
(ii) $\exists\mu$ *such that* $\sup_{x\in\mathcal{A}}\|\dot{y}(t) - G(x)\hat{\dot{y}}(t)\|_2 < \mu, \forall t > 0$;
(iii) $\exists\lambda \in \mathbb{R}_{>0}$ *such that* $(h(x) - h(\hat{x}))^\top G(\hat{x})\mathrm{sign}(h(x) - h(\hat{x})) \geqslant \lambda\|h(x) - h(\hat{x})\|_2$ *for all* $x \in \mathcal{A}_e$, $\tilde{x} \in \Omega_e$.

Thus, for each $d \in \mathbb{R}_{>0}$, there exist $k \in \mathbb{R}_{>0}$ and $T \in \mathbb{R}_{\geqslant 0}$ such that, letting $\hat{x}(t)$ be the solution to system (7), one has $\|\tilde{x}(t)\| := \|x(t) - \hat{x}(t)\| \leqslant d$ for all times $t > T$.

Proof. Consider the function V given in (4). By computing the time derivative of V along the trajectories of system (7) one has that, by items (ii) and (iii),

$$\begin{aligned}\dot{V} &= \tilde{y}^\top\dot{y} - \tilde{y}^\top G(\hat{x})\hat{\dot{y}} - k\|\tilde{y}\|_2^{\alpha}\tilde{y}^\top G(\hat{x})\mathrm{sign}(\tilde{y})\\ &\leqslant \mu\|\tilde{y}\|_2 - k\lambda\|\tilde{y}\|_2^{\alpha+1} \leqslant \|\tilde{y}\|_2(\mu - k\lambda\|\tilde{y}\|_2^{\alpha}).\end{aligned}$$

Therefore, if $\|\tilde{y}\|_2^{\alpha} > \frac{\mu}{\lambda k}$, then $\dot{V}$ is negative, thus implying that there exists $T \in \mathbb{R}_{\geqslant 0}$ such that $\|\tilde{y}(t)\|_2 \leqslant (\frac{\mu}{\lambda k})^{1/\alpha}$ for all $t > T$. Since, by Lemma (L4) of Nicosia et al. (1991b), if (i) holds, then there exists $\gamma \in \mathbb{R}_{\geqslant 0}$ such that $\|\tilde{x}\|_2 < \gamma\|\tilde{y}\|_2$ for any $\tilde{x} \in \Omega_e$ and $x \in \mathcal{A}_e$, if

$$k \geqslant \frac{\gamma\,\mu}{\lambda\,d^{\alpha}},$$

then $\exists T \in \mathbb{R}_{\geqslant 0}$ such that $\|\tilde{x}(t)\| \leqslant d$ for all $t > T$. □

Differently from the observer (3), system (7) provides a finite-time "practical" estimate of $x(t)$, *i.e.*, the estimation error is not zero for all $t > T$, but it can be made arbitrarily small. Note that items (i) and (ii) hold if $\mathcal{A}$ is compact, there is no singular point for h in $\mathcal{A}$, and $\sup_{x\in\mathcal{A}}\{\max\sigma(G(x))\}$, $\|\dot{y}(t)\|_2$, $\|\hat{\dot{y}}(t)\|_2$ are bounded. The following proposition states that item (iii) holds if $\mathcal{A}$ is compact and $\hat{J}(x) \simeq J(x)$ so that $G(x) \simeq I$.

Proposition 1. *If $\|J^{-1}(x)\|_2 \leqslant \kappa_1 < \infty$, $\|\hat{J}^{-1}(x)\|_2 \leqslant \kappa_2 < \infty$, and $\|J(x) - \hat{J}(x)\|_2 < (\sqrt{n}\,\kappa_1\kappa_2)^{-1}$ for all $x \in \mathcal{A}$, then item (iii) of Theorem 2 holds.*

Proof. Let $Z(x) := J^{-1}(x) - \hat{J}^{-1}(x)$. One has that

$$\begin{aligned}&\tilde{y}^\top G(x)\mathrm{sign}(\tilde{y}) = \tilde{y}^\top J(x)(J^{-1}(x) - Z(x))\mathrm{sign}(\tilde{y})\\ &\quad = \|\tilde{y}\|_1 - \tilde{y}^\top Z(x)\mathrm{sign}(\tilde{y}) \geqslant \|\tilde{y}\|_2 - \sqrt{n}\|Z(x)\|_2\|\tilde{y}\|_2.\end{aligned}$$

Therefore, if $\|Z(x)\|_2 < n^{-\frac{1}{2}}$, item (iii) of Theorem 2 holds with $\lambda = 1 - \sqrt{n}\|Z(x)\|_2$. The proof follows by the fact that $\|Z(x)\|_2 = \|J^{-1}(x)(\hat{J}(x) - J(x))\hat{J}^{-1}(x)\|_2 \leqslant \|J^{-1}(x)\|_2\|J(x) - \hat{J}(x)\|_2\|\hat{J}^{-1}(x)\|_2 < n^{-\frac{1}{2}}$. □