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Brief Paper

Constrained control of input–output linearizable systems using control sharing barrier functions☆

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ABSTRACT

Control barrier functions (CBFs) have been used as an effective tool for designing a family of controls that ensures the forward invariance of a set. When multiple CBFs are present, it is important that the set of controls satisfying all the barrier conditions is non-empty. In this paper, we investigate such a control-sharing property for multiple CBFs and provide sufficient and necessary conditions for the property to hold. Based on that, we study the tracking control design problem of an input–output linearizable system with multiple time-varying output constraints, where the output constraints are encoded as CBFs and the barrier conditions are expressed as hard constraints in a quadratic program (QP) whose feasibility is guaranteed by the control-sharing property of the CBFs. With the controller generated from the QP, the output constraints are always satisfied and the tracking objective is achieved when it is not conflicting with the constraints. The effectiveness of our control design method is illustrated by two examples.

1. Introduction

First introduced in optimization, barrier functions (also known as barrier certificates) are now used as an important tool for the verification of nonlinear systems and hybrid systems (Prajna & Jadbabaie, 2004; Prajna, Jadbabaie, & Pappas, 2007; Wisniewski & Sloth, 2016). Using Lyapunov-like conditions, barrier functions can provably establish safety or eventuality properties of dynamical systems without the difficult task of computing the system's reachable set. The extension of barrier functions to a control system results in control barrier functions (CBFs), which, in some sense, parallels the extension of Lyapunov functions to Control Lyapunov function (CLFs) . A family of controls ensuring the forward invariance of a set is established by the barrier condition, which can be used for the control synthesis of systems with state constraints or safety specifications (Ames, Grizzle, & Tabuada, 2014; Panagou, Stipanović, & Voulgaris, 2016; Tee, Ge, & Tay, 2009; Wieland & Allgöwer, 2007).

Depending on the values of a CBF on the associated set, two types of (control) barrier functions are commonly used in literature: one goes to infinity on the set boundary (Ames et al., 2014; Jin & Xu, 2013; Tee et al., 2009), while the other vanishes on the set boundary (Romdlony & Jayawardhana, 2016; Wolff & Buss, 2005; Xu, Tabuada, Ames, & Grizzle, 2015). The former type of CBFs is only defined inside the given set whose boundary cannot be crossed; for example, the reciprocal CBF in Ames et al. (2014), the barrier Lyapunov function (BLF) in Ngo, Mahony, and Jiang (2005) and Tee et al. (2009), and several of its extensions such as the tan-type BLF (Jin, 2017) and the integral BLF (He, Sun, & Ge, 2015). The latter type of CBFs is defined in the whole state space, but the barrier condition ensures that the trajectory of the system will stay inside the set once starting there. Related works belonging to this type include the invariance control (Kimmel & Hirche, 2015; Kimmel, Jahne, & Hirche, 2016; Wolff & Buss, 2005), the control Lyapunov barrier function (Romdlony & Jayawardhana, 2016), and the zeroing CBF (Xu et al., 2015), among others.

Various kinds of barrier conditions have been proposed in literature. A widely used barrier condition for a CBF B is $\dot{B} \leq 0$ (or $\dot{B} \geq 0$ depending on the context), which implies that all the sublevel sets of B are invariant (Prajna et al., 2007; Romdlony & Jayawardhana, 2016; Tee et al., 2009; Wieland & Allgöwer, 2007). Another barrier condition is that given in the invariance control framework, where the higher order derivative condition of a so-called invariance function is implemented such that the function has negative values inside the set. In a recent paper Ames et al. (2014), the barrier condition $\dot{B} \leq 0$ was modified by allowing B to grow when it is far away from the boundary of the set and stop growing when it approaches the boundary. Such a condition enlarges the set of controls that can guarantee the invariance of a given set. CBFs under such a condition are combined with CLFs, which represent

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the performance objectives, in a quadratic program (QP), such that a min-norm control law is generated via real-time optimizations. This idea was further extended in papers such as Ames, Xu, Grizzle, and Tabuada (2017), Nguyen and Sreenath (2016) and Xu et al. (2015), and applied to safety-critical systems (Ames et al., 2017), multi-agent systems (Wang, Ames, & Egerstedt, 2016) and bipedal robots (Hsu, Xu, & Ames, 2015).

When multiple state constraints are presented and each constraint is expressed as a CBF, it is important to ensure that all the barrier conditions can be satisfied simultaneously, that is, the set of controls satisfying all the barrier conditions is non-empty. Particularly, for the QP-based framework proposed in Ames et al. (2014, 2017) and Xu et al. (2015), simultaneous satisfaction of all the barrier conditions is needed to guarantee the feasibility of the QP. Such a shared-control problem has been investigated for CLFs in Andrieu and Prieur (2010), Grammatico, Blanchini, and Andrea (2014) and shown to be hard to solve in general; for instance, it was shown in Grammatico et al. (2014) that two convex CLFs do not necessarily have a common control even for linear time-invariant systems when the dimension of the system is greater than 2.

In this paper, we study the control-sharing property of multiple high order CBFs. Roughly speaking, CBFs are said to have the control-sharing property if for any state, there exists a common control such that the barrier conditions are satisfied simultaneously. Sufficient and necessary conditions for the control-sharing property to hold are given by assuming the CBFs have a well-defined, global relative degree. Based on that, we investigate the tracking control problem for input–output linearizable systems with multiple time-varying output constraints, where each constraint is expressed as a CBF. Sufficient conditions for such CBFs to have the control-sharing property are given. The barrier conditions are expressed as hard constraints in a QP, where the objective function is to minimize the distance between the generated control and a nominal tracking control law. Because of the control-sharing property of the CBFs, the QP is guaranteed to be feasible. Furthermore, the output constraints are always satisfied and the tracking objective is achieved when it is not conflicting with the constraints. Our control design method has several advantages over existing ones, such as the output constraints and the nominal tracking controller can be designed separately, the reference trajectory does not need to be restricted inside the constraint region, and the initial output can be outside the constraint region. Two examples taken from literature are also provided to show the effectiveness of the proposed control design method.

A preliminary version of this work was presented in the conference publication Xu (2016). The present paper is different from Xu (2016) in the following important ways: the input–output linearizable system (instead of the strict-feedback system in Xu, 2016) is considered; a key theorem in Xu (2016) is generalized from the sufficient condition to the sufficient and necessary conditions; the two CBFs case is generalized to the multiple CBFs case. The remainder of the paper is organized as follows. In Section 2, the notion of time-varying control barrier function and the control-sharing property are introduced first, then sufficient and necessary conditions for the control-sharing property to hold are given. In Section 3, the tracking control problem for input–output linearizable systems with multiple output constraints is investigated, where two examples are also provided for illustrative purposes. Finally, some conclusion remarks are given in Section 4.

2. Control-sharing barrier functions

In this section, we first provide a lemma for ensuring non-negativeness of a function through a high order derivative condition, and then introduce the notion of high order, time-varying CBFs. After that, we define the control-sharing property of multiple CBFs and give sufficient and necessary conditions for such a property to hold.

2.1. Control barrier function

Consider a time-varying system

$$\dot{x} = f(t, x), \tag{1}$$

with $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ piecewise continuous in t and locally Lipschitz in x. For any initial condition $x(0)$ at $t = 0$, there exists a maximal time interval $I(x(0))$ such that $x(t)$ is the unique solution to (1). For simplicity, we assume that the system (1) is forward complete, that is, $I(x(0)) = [0, \infty)$.

Given a smooth function $h(t, x) : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$, its first order derivative along the solution of (1) is $h^{(1)}(t, x) = \frac{\mathrm{d}h(t,x)}{\mathrm{d}t} = \frac{\partial h(t,x)}{\partial x} f(t, x) + \frac{\partial h(t,x)}{\partial t}$. The ith($i \geq 2$) order derivative of $h(t, x)$ is computed recursively and denoted as $h^{(i)}(t, x)$. In what follows, we will also use $h^{(i)}$ for $h^{(i)}(t, x)$ when no confusion occurs.

Now suppose that $h(t, x)$ is a C^r function for some positive integer $r \geq 1$ and satisfies the following inequality:

$$h^{(r)} + a_1 h^{(r-1)} + \cdots + a_{r-1} h^{(1)} + a_r h \geq 0, \tag{2}$$

where $a_1, \ldots, a_r \in \mathbb{R}$ are a set of real numbers such that the roots of the polynomial

$$p_0^r(\lambda) = \lambda^r + a_1 \lambda^{r-1} + \cdots + a_{r-1}\lambda + a_r \tag{3}$$

are real numbers $-\lambda_1, \ldots, -\lambda_r$ with $\lambda_i > 0 (1 \leq i \leq r)$. To explore the condition under which $h(t, x)$ is non-negative for $t \geq 0$, we define

$$s_0(t, x) = h(t, x),\ \ s_k = (\frac{\mathrm{d}}{\mathrm{d}t} + \lambda_k)s_{k-1},\ \ 1 \leq k \leq r. \tag{4}$$

It is clear that (2) is equivalent to $s_r(t, x) \geq 0$. Denote $s_k(0, x(0))$ by $s_k(0)$ for short where $k = 0, 1, \ldots, r$. Then, we have the following lemma.

Lemma 1. *Given a $C^r (r \geq 1)$ function $h(t, x) : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ and a set of real numbers $a_1, \ldots, a_r \in \mathbb{R}$ such that $p_0^r(\lambda)$ shown in (3) has roots $-\lambda_1, \ldots, -\lambda_r$ where $\lambda_1, \ldots, \lambda_r > 0$, if s_i defined in (4) satisfy $s_i(0) \geq 0$ for $i = 0, 1, \ldots, r-1$, then $h(t, x) \geq 0$ for any $t \geq 0$.*

Proof. It is clear that inequality (2) is equivalent to $\frac{\mathrm{d}}{\mathrm{d}t}(e^{\lambda_r t} s_{r-1}(t, x(t))) \geq 0$, which results in $s_{r-1}(t, x(t)) \geq s_{r-1}(0)e^{-\lambda_r t}$ by integrating both sides on $[0, t]$. Since $s_{r-1} = (\frac{\mathrm{d}}{\mathrm{d}t} + \lambda_{r-1})s_{r-2}$, we have $\frac{\mathrm{d}}{\mathrm{d}t}(e^{\lambda_{r-1} t} s_{r-2}(t, x(t))) \geq s_{r-1}(0)e^{(\lambda_{r-1}-\lambda_r)t}$. Integrating both sides of this inequality on $[0, t]$ results in $s_{r-2}(t, x(t)) \geq s_{r-1}(0)e^{-\lambda_{r-1}t}\int_0^t e^{(\lambda_{r-1}-\lambda_r)\tau_1}\mathrm{d}\tau_1 + s_{r-2}(0)e^{-\lambda_{r-1}t}$. Continuing this process, we have

$$s_0(t, x(t)) \geq s_0(0)e^{-\lambda_1 t} + \sum_{k=1}^{r-1}[s_k(0)e^{-\lambda_1 t}\int_0^t e^{(\lambda_1-\lambda_2)\tau_k} \int_0^{\tau_k} e^{(\lambda_2-\lambda_3)\tau_{k-1}} \ldots \int_0^{\tau_2} e^{(\lambda_k-\lambda_{k+1})\tau_1}\mathrm{d}\tau_1 \ldots \mathrm{d}\tau_{k-1}\mathrm{d}\tau_k]. \tag{5}$$

For $k = 1, \ldots, r-1$, since $\lambda_i > 0$, it is easy to check that $e^{-\lambda_1 t}\int_0^t e^{(\lambda_1-\lambda_2)\tau_k}\int_0^{\tau_k} e^{(\lambda_2-\lambda_3)\tau_{k-1}} \ldots \int_0^{\tau_2} e^{(\lambda_k-\lambda_{k+1})\tau_1}\mathrm{d}\tau_1 \ldots \mathrm{d}\tau_{k-1}\mathrm{d}\tau_k$ is positive, finite and approaches 0 as $t \to \infty$. Since $s_i(0) \geq 0$ for $i = 0, 1, \ldots, r-1$, the right-hand side of (5) is non-negative, finite and approaches 0 as $t \to \infty$. Therefore, $h(t, x) \geq 0$ for any $t \geq 0$, which completes the proof. □

Remark 1. The conventional comparison lemma cannot be applied to the high order inequality (2) directly (Khalil, 2002). In Gunderson (1971), Gunderson considered the high order differential