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Brief paper

An efficient LQR design for discrete-time linear periodic system based on a novel lifting method☆

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ABSTRACT

This paper proposes a novel lifting method which converts the standard discrete-time linear periodic system to an augmented linear time-invariant system. The linear quadratic optimal control is then based on the solution of the discrete-time algebraic Riccati equation associated with the augmented linear time-invariant model. An efficient algorithm for solving the Riccati equation is derived by using the special structure of the augmented linear time-invariant system. It is shown that the proposed method is very efficient, compared to the ones that use algorithms for discrete-time periodic algebraic Riccati equation. The efficiency and effectiveness of the proposed algorithm is demonstrated by the simulation test for the design problem of spacecraft attitude control using magnetic torques.

1. Introduction

Many engineering systems are naturally periodic, for example, spacecraft attitude control using magnetic torques (Lovera & Astolfi, 2004), helicopter rotors control system (Arcara, Bittanti, & Lovera, 2000), wind turbine control system (Stol, 2003), networked control system (Zhang & Hristu-Varsakelis, 2006), and multirate sampled data system (Khargonekar & Sivashankar, 1991). It has been known for about six decades that linear periodic time-varying system can be converted to some equivalent linear time-invariant systems (Jury & Mullin, 1958, 1959). The most popular and widely used methods that convert the linear periodic time-varying model into linear time-invariant models are the so-called lifting methods proposed in Grasselli and Longhi (1991), Meyer and Burrus (1975) and Varga (2013). These reduced linear time-invariant models are nice for analysis but they are very difficult, if it is not impossible, for Linear Quadratic Regulator (LQR) design. Therefore, the LQR design for linear periodic system has been focused on the periodic system not on the equivalent linear time-invariant systems proposed in Grasselli and Longhi (1991) and Meyer and Burrus (1975). This strategy leads to extensive research on the solutions of the periodic Riccati equations (see Bittanti, 1991; Bittanti, Colaneri, & Guardabassi, 1986; Bittanti, Colaneri, & Nicolao, 1989; Varga, 2008, 2013 and references therein). For the discrete-time linear periodic system, two efficient algorithms for Discrete-time Periodic Algebraic Riccati Equation (DPARE) are emerged (Hench & Laub, 1994; Yang, 2017a).

In this paper, we propose a novel lifting method that converts the linear periodic system to an augmented Linear Time-Invariant (LTI) system. We show that the LQR design method can be directly applied to this LTI system. Moreover, by making full use of the structure of the augmented LTI system, we can derive a very efficient algorithm. We compare the new algorithm to the ones proposed in Hench and Laub (1994) and Yang (2017a). In addition to some simple analysis on the efficiency, we demonstrate the efficiency and effectiveness of the new algorithm by the simulation test for the design problems of spacecraft attitude control using magnetic torques.

The remainder of the paper is organized as follows. Section 2 briefly summarizes the algorithms of Hench and Laub (1994) and Yang (2017a) so that we can compare the proposed algorithm to the existing ones and analyze the efficiency of these algorithms. Section 3 proposes a novel lifting method and applies some standard discrete-time algebraic Riccati equation result to the augmented LTI model. This leads to a very efficient algorithm for the LQR design for the linear periodic system. Section 4 demonstrates the efficiency and effectiveness of the algorithm by some numerical test. Conclusions are summarized in the last section.

2. Periodic LQR design based on linear periodic system

In this section, we briefly review two efficient algorithms for solving DPARE developed in Hench and Laub (1994) and Yang (2017a). This will help us later in the comparison of the proposed method to the existing methods.

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Let p be an integer representing the total number of samples in one period in a periodic discrete-time system. We consider the following discrete-time linear periodic system given as follows:

$$\mathbf{x}_{k+1} = \mathbf{A}_k\mathbf{x}_k + \mathbf{B}_k\mathbf{u}_k, \tag{1}$$

where $\mathbf{A}_k = \mathbf{A}_{k+p} \in \mathbf{R}^{n\times n}$ and $\mathbf{B}_k = \mathbf{B}_{k+p} \in \mathbf{R}^{n\times m}$ are periodic time-varying matrices. For this discrete-time linear periodic system (1), the LQR state feedback control is to find the optimal $\mathbf{u}_k$ to minimize the following quadratic cost function

$$\lim_{N\to\infty}\left(\min \frac{1}{2}\mathbf{x}_N^{\mathrm{T}}\mathbf{Q}_N\mathbf{x}_N + \frac{1}{2}\sum_{k=0}^{N-1}\mathbf{x}_k^{\mathrm{T}}\mathbf{Q}_k\mathbf{x}_k + \mathbf{u}_k^{\mathrm{T}}\mathbf{R}_k\mathbf{u}_k\right) \tag{2}$$

where

$$\mathbf{Q}_k = \mathbf{Q}_{k+p} \geq 0, \tag{3}$$

$$\mathbf{R}_k = \mathbf{R}_{k+p} > 0, \tag{4}$$

and the initial condition $\mathbf{x}_0$ is given. Assume that $\mathbf{Q}_k = \mathbf{C}_k\mathbf{C}_k^{\mathrm{T}}$. It is well-known that under the assumption of the stabilizability of $(\mathbf{A}_k, \mathbf{B}_k)$ and observability of $(\mathbf{A}_k, \mathbf{C}_k)$, the LQR design for problem (1)–(2) can be solved by using the periodic solution of the discrete-time periodic algebraic Riccati equation (Bittanti, 1991). Two efficient algorithms (Hench & Laub, 1994; Yang, 2017a) have been developed to solve p n-dimensional matrix Riccati equations for p positive semidefinite matrices $\mathbf{P}_k$, $k = 1,\ldots,p$. Given $\mathbf{P}_k$, the periodic feedback controllers are given by the following equations:

$$\mathbf{u}_k = -(\mathbf{R}_k + \mathbf{B}_k^{\mathrm{T}}\mathbf{P}_k\mathbf{B}_k)^{-1}\mathbf{B}_k^{\mathrm{T}}\mathbf{P}_k\mathbf{A}_k\mathbf{x}_k. \tag{5}$$

We summarize these two algorithms as follows: Let

$$\mathbf{E}_k = \begin{bmatrix}\mathbf{I} & \mathbf{B}_k\mathbf{R}_k^{-1}\mathbf{B}_k^{\mathrm{T}}\\ \mathbf{0} & \mathbf{A}_k^{\mathrm{T}}\end{bmatrix} = \mathbf{E}_{k+p}, \tag{6}$$

$$\mathbf{F}_k = \begin{bmatrix}\mathbf{A}_k & \mathbf{0}\\ -\mathbf{Q}_k & \mathbf{I}\end{bmatrix} = \mathbf{F}_{k+p}. \tag{7}$$

If $\mathbf{A}_k$ is invertible, then $\mathbf{E}_k$ and $\mathbf{F}_k$ are invertible, and

$$\mathbf{E}_k^{-1} = \begin{bmatrix}\mathbf{I} & -\mathbf{B}_k\mathbf{R}_k^{-1}\mathbf{B}_k^{\mathrm{T}}\mathbf{A}_k^{-\mathrm{T}}\\ \mathbf{0} & \mathbf{A}_k^{-\mathrm{T}}\end{bmatrix} = \mathbf{E}_{k+p}^{-1}$$

and

$$\mathbf{F}_k^{-1} = \begin{bmatrix}\mathbf{A}_k^{-1} & \mathbf{0}\\ \mathbf{Q}_k\mathbf{A}_k^{-1} & \mathbf{I}\end{bmatrix} = \mathbf{F}_{k+p}^{-1}.$$

Let $\mathbf{y}_k$ be the costate of $\mathbf{x}_k$, $\mathbf{z}_k = [\mathbf{x}_k^{\mathrm{T}}, \mathbf{y}_k^{\mathrm{T}}]^{\mathrm{T}}$, and

$$\begin{aligned}\mathbf{\Pi}_k &= \mathbf{E}_{k+p-1}^{-1}\mathbf{F}_{k+p-1}\mathbf{E}_{k+p-2}^{-1}\ldots\mathbf{E}_{k+1}^{-1}\mathbf{F}_{k+1}\mathbf{E}_k^{-1}\mathbf{F}_k\\ &= \mathbf{\Pi}_{k+p},\end{aligned} \tag{8}$$

$$\begin{aligned}\mathbf{\Gamma}_k &= \mathbf{F}_k^{-1}\mathbf{E}_k\mathbf{F}_{k+1}^{-1}\mathbf{E}_{k+1}\ldots,\mathbf{E}_{k+p-2}\mathbf{F}_{k+p-1}^{-1}\mathbf{E}_{k+p-1}\\ &= \mathbf{\Gamma}_{k+p}.\end{aligned} \tag{9}$$

The solutions of p discrete-time periodic algebraic Riccati equations are symmetric positive semi-definite matrices, $\mathbf{P}_k$, $k = 1,\ldots,p$, which are related to the solutions of either one of the two linear systems of equations (Hench & Laub, 1994; Yang, 2017a):

$$\mathbf{z}_{k+p} = \mathbf{\Pi}_k\mathbf{z}_k, \tag{10}$$

$$\mathbf{z}_k = \mathbf{\Gamma}_k\mathbf{z}_{k+p}. \tag{11}$$

Therefore, $\mathbf{P}_k$, $k = 1,\ldots,p$, can be obtained by two methods. The first method uses Schur decomposition:

$$\begin{bmatrix}\mathbf{T}_{11k} & \mathbf{T}_{12k}\\ \mathbf{T}_{21k} & \mathbf{T}_{22k}\end{bmatrix}^{\mathrm{T}}\mathbf{\Pi}_k\begin{bmatrix}\mathbf{T}_{11k} & \mathbf{T}_{12k}\\ \mathbf{T}_{21k} & \mathbf{T}_{22k}\end{bmatrix} = \begin{bmatrix}\mathbf{S}_{11k} & \mathbf{S}_{12k}\\ \mathbf{0} & \mathbf{S}_{22k}\end{bmatrix}, \tag{12}$$

where $\mathbf{S}_{11k}$ is upper-triangular and has all of its eigenvalues inside the unit circle. The periodic solution $\mathbf{P}_k$, $k = 1,\ldots,p$, is given by Hench and Laub (1994)

$$\mathbf{P}_k = \mathbf{T}_{21k}\mathbf{T}_{11k}^{-1}. \tag{13}$$

The second method uses Schur decomposition:

$$\begin{bmatrix}\mathbf{W}_{11k} & \mathbf{W}_{12k}\\ \mathbf{W}_{21k} & \mathbf{W}_{22k}\end{bmatrix}^{\mathrm{T}}\mathbf{\Gamma}_k\begin{bmatrix}\mathbf{W}_{11k} & \mathbf{W}_{12k}\\ \mathbf{W}_{21k} & \mathbf{W}_{22k}\end{bmatrix} = \begin{bmatrix}\mathbf{U}_{11k} & \mathbf{U}_{12k}\\ \mathbf{0} & \mathbf{U}_{22k}\end{bmatrix}, \tag{14}$$

where $\mathbf{U}_{11k}$ is upper-triangular and has all of its eigenvalues outside the unit circle. The periodic solution $\mathbf{P}_k$, $k = 1,\ldots,p$, is given by Yang (2017a)

$$\mathbf{P}_k = \mathbf{W}_{21k}\mathbf{U}_{11k}^{-1}. \tag{15}$$

Remark 2.1. Both algorithms solves general DPARE problem with similar efficiency. But if $\mathbf{A}_k$ and $\mathbf{Q}_k$ are constant matrices, the second method is much efficient because $\mathbf{F}_k$ becomes a constant matrix and $\mathbf{F}_k^{-1} = \cdots = \mathbf{F}_{k+p-1}^{-1} = \mathbf{F}^{-1}$, which makes the computation of (9) much more efficient than the computation of (8).

3. Periodic LQR design based on linear time-invariant system

We propose a lifting method in this section to convert the discrete-time linear periodic system into an augmented linear time-invariant system. Thereby, the periodic LQR design is reduced to the LQR design for the augmented linear time-invariant system.

To simplify our discussion, let us consider a periodic system with $p = 3$. We will use k for the discrete-time in the periodic system and K for the discrete-time in the augmented system.

$$\mathbf{x}_1 = \mathbf{A}_0\mathbf{x}_0 + \mathbf{B}_0\mathbf{u}_0,$$

$$\mathbf{x}_2 = \mathbf{A}_1\mathbf{x}_1 + \mathbf{B}_1\mathbf{u}_1,$$

$$\mathbf{x}_3 = \mathbf{A}_2\mathbf{x}_2 + \mathbf{B}_2\mathbf{u}_2,$$

$$\mathbf{x}_4 = \mathbf{A}_0\mathbf{x}_3 + \mathbf{B}_0\mathbf{u}_3,$$

$$\mathbf{x}_5 = \mathbf{A}_1\mathbf{x}_4 + \mathbf{B}_1\mathbf{u}_4,$$

$$\mathbf{x}_6 = \mathbf{A}_2\mathbf{x}_5 + \mathbf{B}_2\mathbf{u}_5,$$

$$\mathbf{x}_7 = \mathbf{A}_0\mathbf{x}_6 + \mathbf{B}_0\mathbf{u}_6,$$

$$\vdots$$

We can easily regroup the periodic system and rewrite it as the following form:

$$\begin{aligned}\bar{\mathbf{x}}_1 = \begin{bmatrix}\mathbf{x}_1\\ \mathbf{x}_2\\ \mathbf{x}_3\end{bmatrix} &= \begin{bmatrix}\mathbf{0} & \mathbf{0} & \mathbf{A}_0\\ \mathbf{0} & \mathbf{0} & \mathbf{A}_1\mathbf{A}_0\\ \mathbf{0} & \mathbf{0} & \mathbf{A}_2\mathbf{A}_1\mathbf{A}_0\end{bmatrix}\begin{bmatrix}\mathbf{0}\\ \mathbf{0}\\ \mathbf{x}_0\end{bmatrix}\\ &+ \begin{bmatrix}\mathbf{B}_0 & \mathbf{0} & \mathbf{0}\\ \mathbf{A}_1\mathbf{B}_0 & \mathbf{B}_1 & \mathbf{0}\\ \mathbf{A}_2\mathbf{A}_1\mathbf{B}_0 & \mathbf{A}_2\mathbf{B}_1 & \mathbf{B}_2\end{bmatrix}\begin{bmatrix}\mathbf{u}_0\\ \mathbf{u}_1\\ \mathbf{u}_2\end{bmatrix}\\ &= \bar{\mathbf{A}}\bar{\mathbf{x}}_0 + \bar{\mathbf{B}}\bar{\mathbf{u}}_0,\end{aligned}$$

$$\bar{\mathbf{x}}_2 = \begin{bmatrix}\mathbf{x}_4\\ \mathbf{x}_5\\ \mathbf{x}_6\end{bmatrix} = \begin{bmatrix}\mathbf{0} & \mathbf{0} & \mathbf{A}_0\\ \mathbf{0} & \mathbf{0} & \mathbf{A}_1\mathbf{A}_0\\ \mathbf{0} & \mathbf{0} & \mathbf{A}_2\mathbf{A}_1\mathbf{A}_0\end{bmatrix}\begin{bmatrix}\mathbf{x}_1\\ \mathbf{x}_2\\ \mathbf{x}_3\end{bmatrix}$$