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Brief Paper

An iterative algorithm for discrete periodic Lyapunov matrix equations[☆]

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ABSTRACT

In this paper, a novel iterative algorithm with a tuning parameter is developed to solve the forward discrete periodic Lyapunov matrix equation associated with discrete-time linear periodic systems. An important feature of the proposed algorithm is that the information in the current and the last steps is used to update the iterative sequence. The convergence rate of the algorithm can be significantly improved by choosing a proper tuning parameter. It is shown that the sequence generated by this algorithm with zero initial conditions monotonically converges to the unique positive definite solution of the periodic Lyapunov matrix equation if the tuning parameter is within the interval $(0, 1]$. In addition, a necessary and sufficient convergence condition is given for the proposed algorithm in terms of the roots of a set of polynomial equations. Also, a method to choose the optimal parameter is developed such that the algorithm has the fastest convergence rate. Finally, numerical examples are provided to illustrate the effectiveness of the proposed algorithm.

1. Introduction

Periodic linear systems are a class of important time-varying systems, and have found wide applications. For instance, periodic linear systems have been used to capture multirate sampled-date systems (Chen & Francis, 1995), and pendulums (Bittanti, Hernandez, & Zerbi, 1991). In analysis and design of periodic linear systems, the discrete periodic Lyapunov (DPL) matrix equation plays a very important role. In Varga (1997), a necessary and sufficient condition was given for asymptotic stability of the discrete-time periodic linear system in terms of the corresponding DPL matrix equation. In addition, the DPL matrix equation can be also used to check the controllability and observability of discrete-time periodic systems (Halanay & Ionescu, 1994).

Some approaches have been proposed to solve the DPL matrix equation related to the periodic linear system. In Varga (1997), the DPL matrix equation was first transformed into a matrix–vector equation with the help of Kronecker product, and then the solution was explicitly obtained by Doolittle algorithm. An iterative algorithm was also constructed in Varga (1997) in terms of an augmented matrix. It has been known in Sreedhar and Van Dooren (1994) that a DPL matrix equation can be transformed into a standard discrete Lyapunov matrix equation. Hence some methods (Ding & Chen, 2005a, 2005b; Wang, Cheng, & Wei, 2007) for standard Lyapunov matrix equations can be applied to solve DPL matrix equations. A common shortcoming of the previous approaches is that the transformed matrix equations have much higher dimensionality than the original DPL matrix equations. It has been pointed out in Borno (1995) that high-dimensional matrices should be avoided in practical computations.

In Zhou, Duan, and Li (2009), a gradient-based iterative algorithm was developed to solve coupled matrix equations. In Zhang (2015), the reduced-rank gradient-based algorithms were presented for generalized coupled Sylvester matrix equations. The DPL matrix equation is a special coupled matrix equation, and thus the two algorithms in Zhang (2015) and Zhou et al. (2009) can be used to solve it. In Borno and Gajic (1995), a parallel iterative scheme was given for solving coupled Lyapunov matrix equations in discrete-time Markovian jump linear systems. The idea in Borno and Gajic (1995) can be adopted to construct an iterative algorithm for the DPL matrix equation.

When the iterative algorithms in Borno and Gajic (1995), Zhang (2015) and Zhou et al. (2009) are applied to solve the DPL matrix equation, in each iteration step the current estimation of each unknown matrix is updated by only using the information obtained in the last step. In fact, some available information in the last and current steps can be used. This is the idea of "using latest updated

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information", and has been used in Wu and Duan (2015) to construct a new iterative algorithm for solving coupled Lyapunov matrix equations appearing in discrete-time Markovian jump linear systems. Recently, this idea was also applied in Wu and Chang (2016) to obtain iterative algorithms for DPL matrix equations. In this paper, we consider another DPL matrix equation. Based on an equivalent form of the considered DPL matrix equation a novel iterative algorithm is proposed for solving the DPL matrix equation. Analogous to the methods in Wu and Chang (2016) and Wu and Duan (2015), the idea of "using latest updated information" is also used. The convergence property of the proposed algorithm is analyzed in this paper. A necessary and sufficient condition is given for the proposed algorithm to be convergent in terms of the roots of a set of polynomial equations. In addition, an approach is also developed to choose the optimal tuning parameter such that the algorithm achieves the fastest convergence rate.

Throughout this paper, we use A^{T}, $\det(A)$, $\mathrm{tr}(A)$, $\sigma(A)$, and $\rho(A)$ to denote the transpose, determinant, trace, spectrum and spectral radius of the matrix A, respectively. For two integers $a \le b$, the notation $\mathbb{I}[a, b]$ is defined as $\mathbb{I}[a, b] = \{a, a+1, \ldots, b\}$. The notation $\|A\|_{\mathrm{F}}$ refers to the Frobenius norm of the matrix A. $\mathcal{X} = (X_1, X_2, \ldots, X_n)$ denotes a matrix tuple, and $\mathcal{X} > 0$ means all the matrices X_i, $i \in \mathbb{I}[1, n]$, are positive definite. The vectorization of a matrix $A \in \mathbb{R}^{n\times n}$ is defined as $\mathrm{vec}(A) = \begin{bmatrix} a_1^{\mathrm{T}} & a_2^{\mathrm{T}} & \cdots & a_n^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}$ and the notation $A \otimes B$ represents the Kronecker product of matrices A and B. In addition, we use $\mathrm{diag}(A_1, A_2, \ldots, A_n)$ to represent a block diagonal matrix with the diagonal elements $A_1, A_2, \ldots, A_n$.

2. Preliminaries and previous results

Consider the following discrete-time linear periodic system:

$$x(t+1) = A_t x(t), \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $A_t \in \mathbb{R}^{n\times n}$ is the ω-periodic system matrix with $\omega \ge 1$ being an integer, namely,

$$A_{t+\omega} = A_t. \tag{2}$$

The asymptotic stability of the discrete-time linear periodic system (1) can be characterized by the corresponding DPL matrix equation.

Lemma 1 (Varga (1997)). *Let $Q_k > 0$, $k \in \mathbb{I}[1, \omega]$, be given positive definite matrices with $\omega \ge 1$, the linear periodic system (1) is asymptotically stable if and only if the following forward DPL matrix equation*

$$\begin{cases} A_k X_k A_k^{\mathrm{T}} - X_{k+1} = -Q_k, \ \ k \in \mathbb{I}[1, \omega], \\ X_{\omega+1} = X_1, \end{cases} \tag{3}$$

has a unique positive definite solution.

From the conclusion in Sreedhar and Van Dooren (1994), the forward DPL matrix equation (3) can be rewritten as the standard discrete Lyapunov matrix equation $AXA^{\mathrm{T}} - X = -Q$, where

$$\begin{cases} X = \mathrm{diag}\left(X_1, X_2, \ldots, X_\omega\right), \\ Q = \mathrm{diag}\left(Q_\omega, Q_1, \ldots, Q_{\omega-2}, Q_{\omega-1}\right), \\ A = \begin{bmatrix} 0 & A_\omega \\ \mathrm{diag}\left(A_1, A_2, \ldots, A_{\omega-1}\right) & 0 \end{bmatrix}. \end{cases} \tag{4}$$

The gradient based iterative algorithms in Ding and Chen (2005a, 2005b) can be used to solve the matrix equation (3), and the algorithm can be obtained as

$$\begin{cases} X(m+1) = \dfrac{1}{2}[X_1(m+1) + X_2(m+1)], \\ X_1(m+1) = X(m) - \mu A^{\mathrm{T}}[Q + AX(m)A^{\mathrm{T}} - X(m)]A, \\ X_2(m+1) = X(m) + \mu[Q + AX(m)A^{\mathrm{T}} - X(m)], \\ \mu = (\rho^2(AA^{\mathrm{T}}) + 1)^{-1}, \end{cases} \tag{5}$$

where the matrices A and Q are defined in (4), and $X(m)$ is the estimate of the matrix X at the m-th step. By specializing the algorithm in Wang et al. (2007), the following algorithm can be established:

$$\begin{cases} F(m) = Q - X(m) + AX^{\mathrm{T}}(m)A^{\mathrm{T}}, \\ P(m) = F(m) - A^{\mathrm{T}}F^{\mathrm{T}}(m)A, \\ X(m+1) = X(m) + \dfrac{\|F(m)\|_{\mathrm{F}}^2}{\|E(m)\|_{\mathrm{F}}^2} E(m), \\ E(m+1) = P(m+1) - \dfrac{\mathrm{tr}(P^{\mathrm{T}}(m+1)E(m))}{\|E(m)\|_{\mathrm{F}}^2} E(m), \end{cases} \tag{6}$$

where $E(0) = P(0)$, and an arbitrary initial matrix $X(0) \in \mathbb{R}^{n\omega\times n\omega}$ is given.

By using the idea in Borno and Gajic (1995), the following algorithm can be established:

$$X_{k+1}(m+1) = A_k X_k(m) A_k^{\mathrm{T}} + Q_k, \ k \in \mathbb{I}[1, \omega], \tag{7}$$

with $X_{\omega+1}(m) = X_1(m)$ for any integer $m \ge 0$. By using the idea in Borno and Gajic (1995), the convergence condition for the algorithm (7) can be given.

The idea in Wu and Chang (2016) can be used, and thus the following iterative algorithms can be obtained to solve the forward DPL matrix equation (3):

$$\begin{cases} X_k(m+1) = A_{k-1}X_{k-1}(m)A_{k-1}^{\mathrm{T}} + Q_{k-1}, \ k \in \mathbb{I}[1, \omega-1], \\ X_\omega(m+1) = A_{\omega-1}X_{\omega-1}(m+1)A_{\omega-1}^{\mathrm{T}} + Q_{\omega-1}, \end{cases} \tag{8}$$

and

$$\begin{cases} X_\omega(m+1) = A_{\omega-1}X_{\omega-1}(m)A_{\omega-1}^{\mathrm{T}} + Q_{\omega-1}, \\ X_{\omega-k}(m+1) = A_{\omega-k-1}X_{\omega-k-1}(m+1)A_{\omega-k-1}^{\mathrm{T}} \\ \quad + Q_{\omega-k-1}, \ k \in \mathbb{I}[1, \omega-1], \end{cases} \tag{9}$$

with $X_0(m) = X_\omega(m)$, $A_0 = A_\omega$, and $Q_0 = Q_\omega$ for $m \ge 0$.

At the end of this section, we give the following preliminary results, which will be used in the next sections.

Lemma 2 (Bibby (1974)). *Let $\{P(i)\}$ be a sequence of positive definite matrices satisfying the following two properties:*

(1) $P(k) \le P(k+1)$, for any integer $k \ge 0$;

(2) There exists a positive definite matrix P such that $P(i) \le P$ for any integer $i \ge 0$.

Then the sequence $\{P(i)\}$ is convergent.

Lemma 3 (Chen and Chen (2001)). *For any $x(0) \in \mathbb{R}^n$, the sequence $x(m)$ generated by the following iterative algorithm*

$$x(m+1) = Gx(m) + c, \ m \ge 0, \tag{10}$$

converges to the unique solution of $x = Gx + c$ if and only if $\rho(G) < 1$. The convergence rate of this iterative algorithm is

$$R = -\ln \rho(G). \tag{11}$$

Lemma 4 (Chen and Chen (2001)). *Given matrices $B_i \in \mathbb{R}^{n\times n}$, $i \in \mathbb{I}[1, \omega]$, define*

$$\Phi(j) = \left(\prod_{i=\omega-j+1}^{\omega-1} B_{\omega-i}\right)\left(\prod_{i=0}^{\omega-j} B_{\omega-i}\right), j \in \mathbb{I}[2, \omega].$$

Then there holds

$$\begin{bmatrix} 0 & B_\omega \\ \Xi & 0 \end{bmatrix}^\omega = \mathrm{diag}\left(\Phi(1), \Phi(2), \ldots, \Phi(\omega)\right), \tag{12}$$

where $\Xi = \mathrm{diag}\left(B_1, B_2, \ldots, B_{\omega-1}\right)$, and $\Phi(1) = B_\omega B_{\omega-1} \ldots B_1$.