Available online at www.sciencedirect.com

ScienceDirect

IFAC-PapersOnLine 49-6 (2016) 115–120

An Experience of Using LMI Technique in Control Education ⋆

Pavel Pakshin * **Julia Emelianova** *

* *Arzamas Polytechnic Institute of R.E. Alekseev Nizhny Novgorod State Technical University 19, Kalinina Street, Arzamas 607227, Russia (e-mail: pakshinpv@gmail.com)*

Abstract: The LMI technique is very popular in modern control theory and applications due to powerful MATLAB-based LMI solvers and efficient interfaces provided by YALMIP package. This motivates wide application of this technique in control education. Nowadays, there exist good textbooks on the subject written for the graduate level but, in authors' opinion, a short course is needed for the undergraduate level. This course is preceded by the basic course on control theory. In this paper, the authors' experience of developing a short course is described. The sketch and most important issues of this course are also presented.

Keywords: Control education, undergraduate course, linear matrix inequalities, LMI solvers, LQR design.

1. INTRODUCTION

Linear matrix inequalities (LMIs) are a powerful tool for solving many control problems. The theoretical background of the LMI technique is presented in the well-known book Boyd et al. (1994), see also Polyak et al. (2014). This field has long and bright history and is connected with the names of Lyapunov, Yakubovich (the grandfather and father of the field, respectively (see Boyd et al., 1994)), Kalman, Popov, Willems and many other outstanding scientists. The active usage of the LMI technique starts since the 1980s with development of powerful numerical interior point methods for solving LMIs in an efficient manner (Nesterov and Nemirovsky, 1994). Currently, several commercial and noncommercial software packages are available for a simple coding of the general LMI problems and the efficient numerical solution of the typical control problems.

Nowadays, there exist good textbooks on LMIs in systems and control that are intended for a graduate-level course. In particular, consider the well-known course within the graduate program of the Dutch Institute of Systems and Control (DISC)(Scherer and Weiland, 2005). The DISC course has the format of two class hours weakly during the period of eight weeks. At the DISC, the first course on LMIs in control was offered in 1997, when the first draft of this book was distributed as lecture notes among the students. The lecture notes were evaluated by many students yielding much criticism that was taken into account in the next editions.

In authors' opinion, an undergraduate course on LMIs is needed in control education. Indeed, the basic notions of the LMI technique only require academic background in linear algebra and calculus that is given at the undergraduate level (Bachelor program). Based on this fact, the authors have developed a short course within the variative part of the undergraduate curriculum. This course is preceded by a basic course on control theory. Some preliminary motivation of this course was considered in (Pakshin et al., 2012). This paper presents the sketch and most important issues of this course.

2. THE STRUCTURE AND KEY POINTS OF THE COURSE

2.1 Exponential Stability

The course starts with formulation of the exponential stability concept. This concept is clearly defined, representing a global form of stability important for many applications. Consider a nonlinear system described by the state-space model

$$\dot{x}(t) = f(t, x(t)),\ t \geq t_0,\ x(t_0) = x_0, \tag{1}$$

where $x \in \mathbb{R}^n$ denotes the state vector, f is a nonlinear vector function satisfying the Lipschitz condition with a constant L uniformly in t, i.e.,

$$|f(t, x') - f_1(t, x'')| \leq L|x' - x''|,\ x', x'' \in \mathbb{R}^n,$$

and $|\cdot|$ means the Euclidean norm. Also, assume that $f(t, 0) = 0$. In this case, $x(t) \equiv 0$ satisfies (1), yielding the equilibrium state of the system.

Definition 1. The equilibrium state $x(t) \equiv 0$ is said to be exponentially stable if there exist positive numbers α and λ such that

$$|x(t)| \leq \alpha|x(t_0| \exp(-\lambda(t - t_0)). \tag{2}$$

The following theorem by N.N. Krasovskii (Krasovskii, 1963) gives sufficient conditions of exponential stability.

Theorem 1. Suppose there exist a scalar function $V(t, x)$ and positive constants c_1, c_2, c_3 satisfying the inequalities

⋆ This work is performed in framework of contract No. 2.1748.2014/K with Ministry of Education and Science of Russia

10.1016/j.ifacol.2016.07.163

$$c_1|x|^2 \le V(t,x) \le c_2|x|^2, \tag{3}$$

$$\left.\frac{dV(t,x)}{dt}\right|_{(1)} \le -c_3|x|^2, \tag{4}$$

where $\left.\frac{dV(t,x)}{dt}\right|_{(1)} = \frac{\partial V(t,x)}{\partial t}\frac{dx}{dt} = \frac{\partial V(t,x)}{\partial t} f(t,x)$ is the derivative along the trajectories of (1). Then the equilibrium state of the system (1) is exponentially stable.

This theorem can be restated as follows.

Theorem 2. If the system (1) is exponentially stable, then there exists a scalar function $V(t,x)$ satisfying (3) and (4).

As the form of $V(t,x)$ is unknown in the general case, it is difficult to find such a function satisfying Theorems 1 and 2 simultaneously. For the linear time-invariant systems, this function is proved to represent a quadratic form. In this case, the state-space description of (1) is reduced to

$$\dot{x} = Ax, \tag{5}$$

where A is a constant $n \times n$ matrix.

Theorem 3. The equilibrium state of the system (5) is exponentially stable if and only if, for any given positive definite quadratic form $W(x) = x^T Q x$, there exists a positive definite quadratic form $V(x) = x^T P x$ satisfying the equation

$$\left.\frac{dV(x)}{dt}\right|_{(5)} = -W(x). \tag{6}$$

2.2 Lyapunov Equation and Lyapunov Inequality

Equation (6) has a solution if and only if the matrix Lyapunov equation

$$A^T P + PA = -Q \tag{7}$$

has a positive definite solution $P = P^T$ for any given positive definite matrix $Q = Q^T$.

Theorem 4. The Lyapunov equation (7) has a positive definite solution $P = P^T$ for any given positive definite matrix $Q = Q^T$ if and only if the matrix A is a Hurwitz matrix.

The matrix inequality

$$A^T P + PA + Q \preceq 0 \tag{8}$$

is called the Lyapunov inequality. The next Theorem establishes a connection between the Lyapunov equation (7) and the Lyapunov inequality (8).

Theorem 5. The Lyapunov inequality (8) has a positive definite solution $P_i = P_i^T$ if and only if the Lyapunov equation (8) has a positive definite solution $P_i = P_i^T$ and $P_i \succeq P_e$.

2.3 Linear Matrix Inequalities and Convex Analysis

The Lyapunov inequality can be reduced to a special form with respect to the entries of the matrix P. For simplicity, consider a particular case with $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, $P = \begin{bmatrix} x_1 & x_2 \\ x_2 & x_3 \end{bmatrix}$, then

$$A^T P + PA + Q = F(x) = F_0 + x_1 F_1 + x_2 F_2 + x_3 F_3, \tag{9}$$

where $x = [x_1\ x_2\ x_2]^T$, $F_0 = Q$, $F_1 = \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & 0 \end{bmatrix}$, $F_2 = \begin{bmatrix} 2a_{21} & a_{11}+a_{22} \\ a_{11}+a_{22} & 2a_{21} \end{bmatrix}$, $F_3 = \begin{bmatrix} 0 & a_{21} \\ a_{21} & 2a_{22} \end{bmatrix}$. In the general case, let $l = n(n+1)/2$ and $E_1, \ldots, E_l$ be the basis matrices in the space $\mathbb{S}^{n\times n}$ of the symmetric matrices. Then $P = \sum_{i=1}^{l} x_i E_i$ and

$$A^T P + PA + Q = F(x) = F_0 + \sum_{i=1}^{l} x_i F_i, \tag{10}$$

where $F_0 = Q$, $F_i = (A^T E_i + E_i A)$, $i = 1, \ldots, l$. The functions in the right-hand sides of (9) and (10) belong to the class of the functions $F : \mathbb{R}^l \to \mathbb{S}^{n\times n}$ defined by

$$F(x) = F_0 + \sum_{i=1}^{l} x_i F_i, \tag{11}$$

where $F_i = F_i^T \in \mathbb{S}^{n\times n}$, $i = 0, \ldots, l$ are known fixed real symmetric matrices and x_i, $i = 1, \ldots, l$ are scalar variables.

Definition 2. A linear matrix inequality (LMI) in the canonical form is an inequality

$$F(x) \prec 0, \tag{12}$$

where F is defined by (11)

It seems very important to notice here that the linear matrix inequality (12) defines a convex constraint in x. That is, the set

$$\mathcal{S} = \{x | F(x) \prec 0\}$$

of the solutions of the LMI (12) is convex. This fact can be easily checked directly. Let $x_1, x_2 \in \mathcal{S}$ and $\alpha \in (0,1)$; then

$$F(\alpha x_1 + (1-\alpha)x_2) = \alpha F(x_1) + (1-\alpha)F(x_2) \prec 0,$$

since F is affine, $\alpha > 0$ and $(1-\alpha) > 0$. A considerable role in LMI techique is played by the Schur complement lemma.

Lemma 1. Let $F(x)$ be an affine function having the partition

$$F(x) = \begin{bmatrix} F_{11}(x) & F_{12}(x) \\ F_{21}(x) & F_{22}(x) \end{bmatrix} \prec 0,$$

where $F_{11}(x)$ is square. Then $F(x) \prec 0$ if and only if

$$F_{11}(x) \prec 0,$$
$$F_{22}(x) - F_{21}(x)[F_{11}(x)]^{-1}F_{12}(x) \prec 0,$$

or

$$F_{22}(x) \prec 0,$$
$$F_{11}(x) - F_{12}(x)[F_{22}(x)]^{-1}F_{21}(x) \prec 0.$$

It seems very important to fix attention on the following simple lemma, being a key result in robust stability and stabilization subject to the affine or polytopic uncertainties.

Lemma 2. Let $f : \mathcal{S} \to \mathbb{R}$ be a convex function and $\mathcal{S} = \mathrm{co}(\mathcal{S}_0)$. Then $f(x) \le \gamma$ for all $x \in \mathcal{S}$ if and only if $f(x) \le \gamma$ for all $x \in \mathcal{S}_0$.