Automatica 69 (2016) 12–17

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Brief paper

Feedback boundary control of linear hyperbolic systems with relaxation☆

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ARTICLE INFO

Article history:
Received 4 August 2015
Received in revised form
20 December 2015
Accepted 2 February 2016

Keywords:
Stabilization
Hyperbolic relaxation systems
Lyapunov methods
Feedback boundary control

ABSTRACT

We consider boundary stabilization for one-dimensional systems of linear hyperbolic partial differential equations with relaxation structure. Such equations appear in many applications. By combining weighted Lyapunov functions, the structure is used to derive new stabilization results. The result is illustrated with an application to boundary stabilization of water flows in open canals.

1. Introduction

We are interested in boundary stabilization of general hyperbolic PDEs (partial differential equations). Our particular focus is on the influence of the source terms on the design of (dissipative) feedback laws. The control of hyperbolic PDEs has recently gained substantial interest in the mathematical and engineering community due to the wide range of possible applications. Most of the development of the design of suitable boundary feedback control was driven by the St. Venant equations (Bastin, Haut, Coron, & d'Andrea-Novel, 2007; Bedjaoui, Weyer, & Bastin, 2009; de Halleux, Prieur, Coron, d'Andréa Novel, & Bastin, 2003; Diagne, Bastin, & Coron, 2012; Dos Santos Martins, Rodrigues, & Diagne, 2012; Leugering & Schmidt, 2002). Other contributions cover the case of gas dynamics (Gugat & Herty, 2010), traffic flows (Amin, Hante, & Bayen, 2012) or supply chains (Coron & Wang, 2012).

In this paper, we are concerned with a class of hyperbolic PDEs appearing as (intermediate) mathematical models between the Boltzmann equation and hyperbolic conservation laws. They describe various irreversible processes including chemical reactive flows, radiation hydrodynamics, inviscid gas dynamics with relaxation, nonlinear optics, viscoelasticity fluid flows, and many more (Yong, 2008; Zhu, Hong, Yang, & Yong, 2015). The fundamental properties of these physically relevant models have been successfully extracted in Yong (2008, 1999, 2001) (also see Remark 1 in Section 2). These properties are referred to as relaxation structure of hyperbolic PDEs and they will be exploited in the following to investigate exponential stability. The exponential stability will be proven by extending the recently proposed class of Lyapunov functions (Coron, 2007; Coron, Bastin, & d'Andrea-Novel, 2008). To the best of our knowledge, this seems the first place where explicitly the structure is used to prove exponential stability.

We briefly recall other related works in this fields. In Coron (2007, Theorem 13.12), a general result using a smallness assumption on the source terms is given in Coron (2007, Theorem 13.12) or Li (1994). However, this assumption is typically not fulfilled by the previously mentioned mathematical models. In the linear case a weaker condition is proposed in the recent paper (Diagne et al., 2012, Condition C2, Theorem 1). As mentioned in Diagne et al. (2012, Remark 2) it is not straightforward to check whether or not this condition is true. Here we pursue a different approach. We use a modified Lyapunov function exploiting the relaxation structure of the system. As in Diagne et al. (2012) we consider the linear cases with linear boundary conditions. However, we do not require the source term to be marginally diagonally stable as in Diagne et al. (2012, Theorem 2). We also

☆ The work of MH is supported by DFG STE2063/1-1, HE5386/14-1, BMBF KinOpt, and DFG Cluster of Excellence *Integrative Production Technologies in High-Wage Countries*. The work of WaY is supported by NFSC 11471185 and by the Tsinghua Strategic Partnership. The material in this paper was not presented at any conference. This paper was recommended for publication in revised form by Associate Editor Rafael Vazquez under the direction of Editor Miroslav Krstic.

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http://dx.doi.org/10.1016/j.automatica.2016.02.016

refer to Coron et al. (2008); Coron, Vazquez, Krstic, and Bastin (2013), Dos Santos, Bastin, Coron, and d'Andréa Novel (2008), Jeltsema, Ortega, and Scherpen (2004), Krstic and Smyshlyaev (2008), Li (1994), Maschke, Ortega, and van der Schaft (2000) and Zuazua (2006) for related investigations using Lyapunov functions and to Hu, Di Meglio, Vazquez, and Krstic (2015) for recent important results on boundary feedback control of fully general linear coupled systems. All these works do not exploit the physically relevant relaxation structure.

Finally, we apply the result to the Saint Venant Exner model. This is the same example as discussed in Diagne et al. (2012, Section 4). With the new Lyapunov function we could also improve the result presented therein. The motivation to study this particular system lies in the fact that exponential stability is obtained without coordinate transformation and therefore weak requirements on the example (Diagne et al., 2012, Section 4) are possible.

2. Motivation and relaxation structure

Motivated by Diagne et al. (2012) and Yong (2008), we consider a one-dimensional linear system

$$u_t + a\, u_x + b\, q_x = 0, \qquad q_t + c\, u_x + d\, q_x = -e\, q \tag{1}$$

for $x \in [0, 1]$ and $t \geq 0$. Here $u : [0, \infty) \times [0, 1] \to \mathbb{R}^{n-r}$, $q : [0, \infty) \times [0, 1] \to \mathbb{R}^r$ and $A := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{R}^{n\times n}$, $e \in \mathbb{R}^{r\times r}$. Unlike that in Diagne et al. (2012), system (1) is *not* in its characteristic form, rather than in its standard form (Yong, 2001).

About this system, we make the following assumptions.

(A1) There exists a symmetric positive definite matrix $A_0 \in \mathbb{R}^{n\times n}$ such that

$A_0 A$ is symmetric and $A_0 = \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix}$

with $X_1 \in \mathbb{R}^{(n-r)\times(n-r)}$ and $X_2 \in \mathbb{R}^{r\times r}$.

(A2) The matrix

$X_2\, e + e^T\, X_2$ is positive definite.

(A3) The coefficient matrix A has no vanishing eigenvalues.

In (A2), the superscript ^{T} has been used in e^T to denote the transpose of the matrix e.

Remark 1. Assumptions (A1) and (A2) are exactly the second structural stability condition proposed in Yong (1999) for general $n \times n$ system $U_t + AU_x = QU$: There exist an invertible matrix $\bar{P} \in \mathbb{R}^{n\times n}$ and an invertible matrix $S \in \mathbb{R}^{r\times r}$ such that $\bar{P} Q \bar{P}^{-1} = \begin{pmatrix} 0 & 0 \\ 0 & S \end{pmatrix}$; there exists a symmetric positive definite matrix $\bar{A}_0$ such that $\bar{A}_0 A$ is symmetric; and

$$\bar{A}_0 Q + Q^T \bar{A}_0 \leq -\bar{P}^T \begin{pmatrix} 0 & 0 \\ 0 & Id_{r\times r} \end{pmatrix} \bar{P}.$$

As shown in Yong (2008, 1999), such conditions are fulfilled by many classical models from mathematical physics. They ensure existence of the zero-relaxation limit for initial-value problems of general multi-dimensional nonlinear systems.

Assumption (A1) implies that the system (1) is hyperbolic. Thus, we can diagonalize the coefficient matrix A with a transformation matrix $\mathbf{T} \in \mathbb{R}^{n\times n}$ such that

$$\mathbf{T}^{-1} A \mathbf{T} = \Lambda, \quad \Lambda := \begin{pmatrix} \Lambda_+ & 0 \\ 0 & \Lambda_- \end{pmatrix}, \qquad \begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix} = \mathbf{T}^{-1} \begin{pmatrix} u \\ q \end{pmatrix}. \tag{2}$$

Here $\Lambda_\pm$ are diagonal and contain the positive and negative eigenvalues of A, respectively; and $\xi_+ \in \mathbb{R}^m$ with m the number of positive eigenvalues of A and therefore $\xi_- \in \mathbb{R}^{n-m}$.

Boundary conditions are specified as

$$\xi_+(t, 0) = K_{00}\xi_+(t, 1) \quad \text{and} \quad \xi_-(t, 1) = K_{11}\xi_-(t, 0). \tag{3}$$

In addition, Eqs. (1) are accompanied by suitable initial data

$$u(0, x) = u_0(x) \quad \text{and} \quad q(0, x) = q_0(x). \tag{4}$$

Remark 2. More general conditions of the type

$$\begin{pmatrix} \xi_+(t, 0) \\ \xi_-(t, 1) \end{pmatrix} = \begin{pmatrix} K_{00} & K_{01} \\ K_{10} & K_{11} \end{pmatrix} \begin{pmatrix} \xi_+(t, 1) \\ \xi_-(t, 0) \end{pmatrix}$$

have been considered in Coron et al. (2008) and Diagne et al. (2012). However, our focus is the treatment of the relaxation term and therefore only consider the simplified setting of Eq. (3).

Assumptions (A1) and (A2) guarantee exponential decay in q. Our goal is to prescribe a feedback boundary control yielding also exponential decay in the conservative variable u. It is known that for $(u_0, q_0) \in L^2((0, 1); \mathbb{R}^n)$ the problem (1) together with (3) and (4) has a unique weak solution $(u, q)(t, \cdot) \in L^2((0, 1); \mathbb{R}^n)$ (Coron, d'Andrea Novel, & Bastin, 2007, Sec 2.1).

Definition 1. The system (1) together with (3) and (4) is exponentially stable, if there exist $\nu > 0$ and $C > 0$ such that, for every $(u_0, q_0) \in L^2((0, 1); \mathbb{R}^n)$, the weak solution to Eqs. (1) together with (3) and (4) satisfies

$$\|(u, q)(t, \cdot)\|_{L^2((0,1);\mathbb{R}^n)} \leq C \exp(-\nu t) \|(u_0, q_0)\|_{L^2((0,1);\mathbb{R}^n)}.$$

In Diagne et al. (2012, Theorem 2) the authors prove exponential stability under the assumption that the matrix $M := \mathbf{T}^{-1} \begin{pmatrix} 0 & 0 \\ 0 & -e \end{pmatrix} \mathbf{T}$ is diagonally marginally stable, i.e., there exists a diagonal positive definite matrix P such that $M^T P + PM$ is negative semi-definite. Unfortunately, it seems *a priori* not clear if such a matrix P exists. Further, its construction might be difficult. Here we exploit the physically relevant assumptions (A1) and (A2) to obtain exponential stability without any further requirements.

Notation: $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ denote the smallest and largest eigenvalue of a matrix A, respectively. To simplify the notation we set $\xi(t) := \xi(t, \cdot) \in L^2((0, 1); \mathbb{R}^n)$ and write $\|\xi(t)\|_A^2 = \int_0^1 \xi^T(t, x) A \xi(t, x) dx$ for a positive definite matrix A. We drop the subindex if the usual L^2-scalar product is used. Clearly, $\lambda_{\min}(A)\|\xi(t)\|^2 \leq \|\xi(t)\|_A^2 \leq \lambda_{\max}(A)\|\xi(t)\|^2$.

3. A modified Lyapunov function for exponential decay

Our main result reads as

Theorem 3.1. *Suppose the system (1) fulfills the assumptions* (A1), (A2) *and* (A3). *Then there exist K_{00} and K_{11} such that the system (1) together with (3) and (4) is exponentially stable.*

The key for the proof of Theorem 3.1 is the choice of an appropriate Lyapunov function. Here we write $U := \begin{pmatrix} u \\ q \end{pmatrix} \in \mathbb{R}^n$ and choose

$$\begin{aligned} \mathcal{L}(t) &= \int_0^1 U^T (\alpha A_0 + \mu(x))\, U dx \\ &= \alpha \|(u, q)(t)\|_{A_0}^2 + \|(u, q)(t)\|_\mu^2 \end{aligned} \tag{5}$$

for some $\alpha > 0$ and

$$\mu(x) := (\mathbf{T}^{-1})^T \exp(-\Lambda x) \mathbf{T}^{-1} \tag{6}$$

with $\mathbf{T}$ and Λ given in Eq. (2). We denote by $\exp(-\Lambda x)$ the diagonal matrix with entries $\exp(-\Lambda_{ii} x)$ for $i = 1, \ldots, n$. Note that $\mu(x)$