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Brief paper

Constraint generalized Sylvester matrix equations☆

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ABSTRACT

In this paper, some necessary and sufficient conditions are established for the constraint generalized Sylvester matrix equations to have a common solution. The expression of the general common solution is also given under the solvable conditions. In addition, a numerical example is presented to illustrate the presented theory.

1. Introduction

In this paper, we denote the complex number field by $\mathbb{C}$. The set of all matrices of dimension $m \times n$ is designated by $\mathbb{C}^{m\times n}$. I denotes an identity matrix having appropriate dimension. For a complex matrix A, the symbols A^* and $r(A)$ stand for the conjugate transpose and rank of A, respectively. The Moore–Penrose inverse of $A \in \mathbb{C}^{m\times n}$, denoted by $A^\dagger$, is defined to be the unique solution X to the following four matrix equations

$$AXA = A, \qquad XAX = X, \qquad (AX)^* = AX, \qquad (XA)^* = XA.$$

Furthermore, L_A and R_A stand for the two projectors $L_A = I - A^\dagger A$ and $R_A = I - AA^\dagger$ induced by A, respectively. It is obvious that

$$R_A = (R_A)^2 = (R_A)^* = R_A^\dagger,$$
$$L_A = (L_A)^* = (L_A)^2 = L_A^\dagger.$$

The Sylvester matrix equation $AX - XB = C$ or generalized Sylvester matrix equation $AX - YB = C$ has massive applications in control theory (Wimmer, 1994; Wu, Duan, & Xue, 2007; Wu, Duan, & Zhou, 2008), H_α-optimal control (Saberi, Stoorvogel, & Sannuti, 2003), linear descriptor systems (Darouach, 2006), sensitivity analysis (Barraud, Lesecq, & Christov, 2001), perturbation theory (Li, 1999), system design (Syrmos & Lewis, 1994) and singular system control (Shahzad, Jones, Kerrigan, & Constantinides, 2011). The use of Sylvester and $*$-Sylvester matrix equations in the disciplines of theory of orbits can be found in Terán and Dopico (2011).

Recently, some mixed Sylvester matrix equations were investigated in some papers. Lee and Vu (2012) gave some solvability conditions to mixed Sylvester matrix equations

$$\begin{aligned} A_1X - YB_1 &= C_1, \\ A_2Z - YB_2 &= C_2. \end{aligned} \tag{1}$$

The researchers proved that the mixed Sylvester matrix equations (1) are consistent if and only if there exist invertible matrices P_1, P_2 and Q such that

$$\begin{bmatrix} A_1 & C_1 \\ 0 & B_1 \end{bmatrix} P_1 = Q \begin{bmatrix} A_1 & 0 \\ 0 & B_1 \end{bmatrix},$$
$$\begin{bmatrix} A_2 & C_2 \\ 0 & B_2 \end{bmatrix} P_2 = Q \begin{bmatrix} A_2 & 0 \\ 0 & B_2 \end{bmatrix}.$$

The general solution to (1) was established by Wang and He (2013). Wang and He (2014) considered some systems of coupled generalized Sylvester matrix equations.

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Motivated by the work mentioned above and keeping the interests and wide applications of generalized Sylvester matrix equations, we consider constraint generalized Sylvester matrix equations:

$$\begin{array}{ll} A_3X = C_3, & YB_3 = C_4, \\ A_4Z = C_5, & A_5ZB_5 = C_6, \\ A_1X - YB_1 = C_1, & A_2Z - YB_2 = C_2, \end{array} \tag{2}$$

which is a more general form of the generalized Sylvester matrix equation $AX - YB = C$ and the mixed Sylvester matrix equations (1). Solving system (2) will improve the theoretical advancement of the mixed Sylvester matrix equations (1).

The principal task of this paper is to establish some necessary and sufficient conditions and the expression of the general solution to (2) when it is consistent.

The remainder of this paper is composed as follows. In Section 2, we present some necessary and sufficient conditions for (2) to have a solution and its exclusive expression is also constructed when solvable conditions are satisfied. In Section 3, an algorithm and a numerical example are given to exemplify our key result. Conclusion is presented in Section 4.

2. Investigation to the system (2)

We commence from some known results. Notice that

$$A_1U + VB_1 + C_3WD_3 + C_4ZD_4 = E_1 \tag{3}$$

can play an important role in the construction of the solution to (2).

Lemma 2.1 (Wang & He, 2012). *Let $A_1, B_1, C_3, D_3, C_4, D_4$ and E_1 be known. Set*

$$A = R_{A_1}C_3, \quad B = D_3L_{B_1}, \quad C = R_{A_1}C_4, \quad D = D_4L_{B_1},$$
$$E = R_{A_1}E_1L_{B_1}, \quad F = R_AC, \quad G = DL_B, \quad H = CL_F.$$

Then Eq. (3) has a solution if and only if

$$R_FR_AE = 0, \quad EL_BL_G = 0, \quad R_AEL_D = 0, \quad R_CEL_B = 0.$$

Under these conditions, the general solution to (3) is

$$U = A_1^{\dagger}(E_1 - C_3WD_3 - C_4ZD_4) - A_1^{\dagger}S_7B_1 + L_{A_1}S_6,$$
$$V = R_{A_1}(E_1 - C_3WD_3 - C_4ZD_4)B_1^{\dagger} + A_1A_1^{\dagger}S_7 + S_8R_{B_1},$$
$$W = A^{\dagger}EB^{\dagger} - A^{\dagger}CF^{\dagger}EB^{\dagger} - A^{\dagger}HC^{\dagger}EG^{\dagger}DB^{\dagger}$$
$$\quad - A^{\dagger}HS_2R_GDB^{\dagger} + L_AS_4 + S_5R_B,$$
$$Z = F^{\dagger}ED^{\dagger} + H^{\dagger}HC^{\dagger}EG^{\dagger} + L_FL_HS_1 + L_FS_2R_G + S_3R_D,$$

where $S_1, \ldots, S_8$ are arbitrary matrices over $\mathbb{C}$ with appropriate sizes.

Lemma 2.2 (Baksalary & Kala, 1979). *Known E, F and G matrices over $\mathbb{C}$ of adequate dimensions, $EX - YF = G$ has a solution if and only if $R_EGL_F = 0$. With this condition, its explicit solution is*

$$X = E^{\dagger}G + W_1F + L_EW_2,$$
$$Y = -R_EGF^{\dagger} + EW_1 + W_3R_F,$$

where W_1, W_2 and W_3 are arbitrary matrices over $\mathbb{C}$ with appropriate sizes.

Lemma 2.3 (Marsaglia & Styan, 1974). *Let $K \in \mathbb{C}^{m\times n}$, $P \in \mathbb{C}^{m\times t}$, $Q \in \mathbb{C}^{l\times n}$. Then*

$$r\begin{bmatrix} K \\ Q \end{bmatrix} = r(QL_K) + r(K),$$
$$r\begin{bmatrix} K & P \end{bmatrix} = r(R_PK) + r(P),$$
$$r\begin{bmatrix} K & P \\ Q & 0 \end{bmatrix} = r(R_KPL_Q) + r(P) + r(Q).$$

Lemma 2.4 (Buxton, Churchouse, & Tayler, 1990). *Let A_1 and C_1 be known matrices with allowable dimensions. Then $A_1X = C_1$ has a solution if and only if $R_{A_1}C_1 = 0$. In this term, its general solution is*

$$X = A_1^{\dagger}C_1 + L_{A_1}T,$$

where T is an arbitrary matrix over $\mathbb{C}$ with appropriate size.

Lemma 2.5 (Buxton et al., 1990). *Let B_1 and D_1 be known matrices with feasible dimensions. Then $YB_1 = D_1$ has a solution if and only if $D_1L_{B_1} = 0$. Under this condition, its general solution is*

$$Y = D_1B_1^{\dagger} + SR_{B_1},$$

where S is an arbitrary matrix over $\mathbb{C}$ with appropriate size.

Lemma 2.6 (Wang, 2005). *Let A_4, A_5, B_5, C_5 and C_6 be given matrices over $\mathbb{C}$ with able dimensions. Set $A_6 = A_5L_{A_4}$. Then the following statements are equivalent:*

(1) *The system of matrix equations $A_4Z = C_5$, $A_5ZB_5 = C_6$ is consistent.*

(2)

$$R_{A_4}C_5 = 0, \quad R_{A_6}(C_6 - A_5A_4^{\dagger}C_5B_5) = 0, \quad C_6L_{B_5} = 0.$$

(3)

$$r[A_4\ C_5] = r(A_4), \quad r\begin{bmatrix} A_4 & C_5B_5 \\ A_5 & C_6 \end{bmatrix} = r\begin{bmatrix} A_4 \\ A_5 \end{bmatrix},$$
$$r\begin{bmatrix} C_6 \\ B_5 \end{bmatrix} = r(B_5).$$

With these conditions, its general solution is

$$Z = A_4^{\dagger}C_5 + L_{A_4}A_6^{\dagger}(C_6 - A_5A_4^{\dagger}C_5B_5)B_5^{\dagger} + L_{A_4}L_{A_6}Q_1 + L_{A_4}Q_2R_{B_5},$$

where Q_1 and Q_2 are arbitrary matrices over $\mathbb{C}$ with appropriate sizes.

Now we demonstrate the main Theorem of this paper.

Theorem 2.1. *Given $A_1, A_2, A_3, A_4, A_5, B_1, B_2, B_3, B_5, C_1, C_2, C_3, C_4, C_5$ and C_6 of fit dimensions over $\mathbb{C}$. Set*

$$A_6 = A_5L_{A_4}\ A_7 = A_1L_{A_3}, \quad B_7 = R_{B_3}B_1,$$
$$C_9 = C_1 - A_1A_3^{\dagger}C_3 + C_4B_3^{\dagger}B_1, \quad A_8 = A_2L_{A_4}L_{A_6},$$
$$B_8 = R_{B_7}R_{B_3}B_2, \quad A_9 = -A_7, \quad B_9 = R_{B_3}B_2, \quad B_{10} = R_{B_5},$$
$$E_1 = C_2 - A_2[A_4^{\dagger}C_5 + L_{A_4}A_6^{\dagger}(C_6 - A_5A_4^{\dagger}C_5B_5)B_5^{\dagger}]$$
$$\quad + [C_4B_3^{\dagger} - R_{A_7}C_9B_7^{\dagger}R_{B_3}]B_2,$$

$$A_{10} = A_2L_{A_4}, \quad A = R_{A_8}A_9, \quad B = B_9L_{B_8}, \tag{4}$$

$$C = R_{A_8}A_{10}, \quad D = B_{10}L_{B_8}, \quad E = R_{A_8}E_1L_{B_8}, \tag{5}$$

$$F = R_AC, \quad G = DL_B, \quad H = CL_F. \tag{6}$$

Then the following statements are equivalent:

(1) *The system (2) has a solution.*

(2)

$$\begin{array}{lll} R_{A_3}C_3 = 0, & C_4L_{B_3} = 0, & R_{A_4}C_5 = 0, \\ C_6L_{B_5} = 0, & R_{A_6}(C_6 - A_5A_4^{\dagger}C_5B_5) = 0, & \\ R_{A_7}C_9L_{B_7} = 0, & R_FR_AE = 0, & EL_BL_G = 0, \\ R_AEL_D = 0, & R_CEL_B = 0. & \end{array} \tag{7}$$

(3)

$$r\begin{bmatrix} A_3 & C_3 \end{bmatrix} = r(A_3), \quad r\begin{bmatrix} C_4 \\ B_3 \end{bmatrix} = r(B_3), \tag{8}$$

$$r\begin{bmatrix} A_4 & C_5 \end{bmatrix} = r(A_4), \tag{9}$$