Automatica 69 (2016) 308–314

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Brief paper

Continuous terminal sliding-mode controller ☆

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ARTICLE INFO

Article history:
Received 13 February 2014
Received in revised form
17 January 2016
Accepted 26 January 2016

Keywords:
Third-order sliding-mode
Finite time stability
Lyapunov stability

ABSTRACT

For uncertain systems with relative degree two, a continuous homogeneous sliding-mode control algorithm is proposed. This algorithm ensures finite-time convergence to the third-order sliding set, using only information about the output and its first derivative. We prove the convergence of the proposed algorithm via a homogeneous, continuously differentiable and strict Lyapunov function.

1. Introduction

Sliding-mode control (SMC) (Edwards & Spurgeon, 1998; Utkin, Guldner, & Shi, 2009) is one of the most efficient control techniques for controlling plants under heavy uncertainty conditions. The goal of such controllers is to (theoretically exactly) compensate a matched uncertainty by keeping some properly chosen (sliding) variables at zero. To reach this goal, theoretically infinite switching frequency is required (Utkin et al., 2009). Such control is not desirable from the implementation point of view due to the oscillations caused by the high-frequency switching, which results in dangerous system vibrations (chattering) (see e.g. Boiko, 2009, Edwards & Spurgeon, 1998, Utkin et al., 2009).

In the last two decades, a number of methods have been proposed to alleviate the chattering effect (see Fridman, 2011 and references cited therein). One of the most popular methods is the higher-order sliding modes (HOSM) approach (Bartolini, Ferrara, & Usai, 1998; Levant, 1993, 2003). HOSM algorithms, for systems with relative degree r, ensure finite-time convergence to zero of the output σ and its first $(r-1)$ derivatives. The asymptotic accuracy of the rth HOSM was analyzed in Levant (1993), where it was shown that the best possible asymptotic accuracy with e.g. sampling interval τ is $\sigma^{(j)} = O\left(\tau^{r-j}\right)$, $j = 0, 1, \ldots, r-1$. Homogeneous HOSM controllers of order r provide this accuracy (Levant, 2005a), so that they have this optimal precision of the output tracking with respect to the sampling step, measurement noises, and fast actuators' dynamics (Levant, 1993, 2003, 2005b; Levant & Fridman, 2010). The main drawback is that the control signal is discontinuous, which produces chattering.

The super-twisting algorithm (STA) plays a special role among the sliding-mode controllers. In contrast to the other HOSM algorithms, it was designed for systems with relative degree one with respect to the output σ, and has the following advantages: (i) it compensates matched Lipschitz uncertainties/perturbations, while first-order sliding-mode can compensate discontinuous and uniformly bounded uncertainties/perturbations; (ii) it provides finite-time convergence to $\sigma = \dot{\sigma} = 0$ simultaneously; (iii) it requires only the information of the output σ; (iv) it generates a continuous control signal and, consequently, reduces chattering effects; (v) it has sliding accuracy of order one with respect to $\dot{\sigma}$ and two with respect to σ.

The first proof of the convergence of the STA was based on the idea of majorant curves (Levant, 1993, 1998). Later, proofs

☆ The authors want to acknowledge the joint scientific program Mexico-India: Conacyt, Project 193564, and DST project DST/INT/MEX/RPO-02/2008. This work was also supported in part by DGAPA-UNAM, projects PAPIIT IN113216 and IN113614, Conacyt projects 241171 and 261737, and Fondo de Cooperación II-FI, Project IISGBAS-100-2015. The material in this paper was not presented at any conference. This paper was recommended for publication in revised form by Associate Editor Michael V. Basin under the direction of Editor Ian R. Petersen.

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http://dx.doi.org/10.1016/j.automatica.2016.02.001
0005-1098/Published by Elsevier Ltd.

based on Lyapunov functions were found (Moreno & Osorio, 2008, 2012; Orlov, Aoustin, & Chevallereau, 2011; Polyakov & Poznyak, 2009). One can apply the STA to alleviate the chattering in systems with relative degree two (see for example Fridman, 2011 and references therein). It is natural in this case to select the sliding variable as $s = \dot{\sigma} + c\sigma, c > 0$. With this choice of s, the STA ensures the uncertainties/perturbations compensation, finite-time convergence to the set $s(t) = \dot{s}(t) = 0$, but the states σ, and $\dot{\sigma}$ converge only *asymptotically* to the origin.

Arbitrary order sliding mode approaches (Bartolini et al., 1998; Levant, 2003, 2005b) can also be used to attenuate the chattering in the control of systems with relative degree two with respect to σ. In order to adjust the chattering and, at the same time generate a continuous signal, one rises artificially the relative degree to three by obtaining $\ddot{\sigma}$, and uses third-order nested or quasi-continuous SMC on the new control variable $\dot{u}$. This ensures the *finite-time* convergence to the origin of σ and $\dot{\sigma}$ and compensates Lipschitz perturbations. The disadvantage of this approach is that the values of $\ddot{\sigma}$ are needed in order to realize third order SMC (Levant, 2003, 2005b). For systems with known control gain the knowledge of $\ddot{\sigma}$ means the knowledge of the perturbations or uncertainties. In this case the perturbations/uncertainties could be compensated directly.

Therefore, the problem of designing a controller which is continuous and, at the same time, ensures finite-time convergence to the third-order sliding-mode, using only information about $\sigma, \dot{\sigma}$, is an important task. The first results solving this problem were obtained in Basin and Rodriguez-Ramirez (2014), Edwards and Shtessel (2014), Moreno, Negrete, Torres-González, and Fridman (2015), Torres González, Fridman, and Moreno (2015) and Zamora, Moreno, and Kamal (2013), where a combination of the super-twisting and twisting or continuous algorithms were reported, ensuring finite-time convergence of $\sigma, \dot{\sigma}$ to the origin.

The goal of this paper is to propose a homogeneous control algorithm for uncertain second-order plants, having the following advantages:

- it compensates Lipschitz uncertainties/perturbations;
- it provides finite-time convergence to third-order sliding-mode set, and therefore provides sliding accuracy of order *three* with respect to σ;
- it generates a continuous control signal;
- it requires only the knowledge of the output σ and $\dot{\sigma}$.

This algorithm can be considered as a combination of super-twisting and terminal sliding-mode. This result has been announced in Fridman, Moreno, Bandyopadhyay, Kamal, and Chalanga (2015, Section 5.2) as Continuous Nonsingular Terminal Sliding Mode Algorithm, but without providing a convergence proof, which is the main contribution of the present paper.

In this light we denote the proposed controller as *Continuous Terminal Sliding Mode* (CTSM) Controller. To prove the finite-time convergence and robustness properties of the CTSM algorithm we use a continuously-differentiable, homogeneous and strict Lyapunov function.

1.1. Notation and definitions

In this paper the following notation is used: $\lceil z \rfloor^p = |z|^p sign(z)$ where $z \in \mathbb{R}$ and $p \in \mathbb{R}$. Therefore $\lceil z \rfloor^2 = |z|^2 sign(z) \neq z^2$. Note that, if p is an odd number, the two expressions $\lfloor z \rceil^p = z^p$ are equivalent. In particular, $\lceil z \rfloor^0 = sign(z)$, $\lceil z \rfloor^0 |z|^p = \lceil z \rfloor^p$, $\lceil z \rfloor^p \lceil z \rfloor^q = |z|^{p+q}$.

Homogeneous functions and systems have appealing properties and they play an important role in HOSM and in this paper. Classical homogeneity corresponds to the scaling property of a scalar function $f(kx) = k^\delta f(x)$, for all $k > 0$ and some $\delta \in \mathbb{R}$. From Bacciotti and Rosier (2005) we recall some definitions of (weighted) homogeneous functions and vector fields and, from Levant (2005a) and Orlov (2003), the corresponding concepts for vector-set fields. A function $f : \mathbb{R}^n \to \mathbb{R}$ (or a multivalued function $F : \mathbb{R}^n \rightrightarrows \mathbb{R}$, $F(x) \subset \mathbb{R}$) is called *homogeneous of degree* δ *with the dilation* $d_k : (x_1, x_2, \ldots, x_n) \mapsto (k^{r_1}x_1, k^{r_2}x_2, \ldots, k^{r_n}x_n)$, where $r_1, \ldots, r_n$ are some positive numbers (called *weights*) if, for any $k > 0$, the identity $f(d_k x) = k^\delta f(x)$ holds (respectively, $F(d_k x) = k^\delta F(x)$).

A vector field $f : \mathbb{R}^n \to \mathbb{R}^n$ (or a vector-set field $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$, $F(x) \subset \mathbb{R}^n$) is homogeneous of degree δ if for any $k > 0$ the identity $f(d_k x) = k^\delta d_k f(x)$ holds (respectively, $F(d_k x) = k^\delta d_k F(x)$). If a vector field has homogeneity degree $\delta \neq 0$, then it can be always scaled to ± 1 by an appropriate proportional change of the weights $r_1, \ldots, r_n$.

2. Problem statement and main result

We consider a perturbed second-order plant described as

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= u + \mu(t), \end{aligned} \tag{1}$$

where $x_1, x_2 \in \mathbb{R}$ are the states, $u \in \mathbb{R}$ is the control and $\mu(t)$ is the perturbation, which is a Lipschitz continuous time signal with Lipschitz constant Δ, i.e. $|\dot{\mu}(t)| \leq \Delta$ almost everywhere. The problem is to design a (time) continuous control law such that the output $\sigma = x_1$ and its derivative $\dot{\sigma} = x_2$ converge in finite time and remain in zero $\sigma = \dot{\sigma} \equiv 0$ despite of the perturbation $\mu(t)$. Moreover, after a finite time the control should compensate for the perturbation, i.e. $u(t) \equiv -\mu(t)$, so that $\ddot{\sigma} \equiv 0$ also, establishing a third order sliding mode.

The problem is solved by the dynamic feedback control law

$$\begin{aligned} u &= -k_1 L^{\frac{2}{3}} \lceil \phi_L(x_1, x_2) \rfloor^{\frac{1}{3}} + z \\ \dot{z} &= -k_2 L \lceil \phi_L(x_1, x_2) \rfloor^0, \end{aligned} \tag{2}$$

where

$$\phi_L(x_1, x_2) = x_1 + \frac{\alpha}{L^{\frac{1}{2}}} \lceil x_2 \rfloor^{\frac{3}{2}}, \tag{3}$$

is a continuously differentiable function of the state. In turn, the parameters k_i and L are positive gains to be designed. When $L = 1$ we denote $\phi_1(x_1, x_2)$ simply as $\phi(x_1, x_2)$. By defining $x_3 \triangleq z + \mu$, the closed loop (1)–(2) yields the (discontinuous) 3rd-order differential equation

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -k_1 L^{\frac{2}{3}} \lceil \phi_L(x_1, x_2) \rfloor^{\frac{1}{3}} + x_3 \\ \dot{x}_3 &= -k_2 L \lceil \phi_L(x_1, x_2) \rfloor^0 + \dot{\mu}(t). \end{aligned} \tag{4}$$

Since the right hand side of (4) is discontinuous, its solutions will be understood in the sense of Filippov (Filippov, 1988). Notice that the Filippov differential inclusion corresponding to the discontinuous and uncertain system (4) is homogeneous of degree (scaled to) $\delta = -1$ and has weights $r = [3, 2, 1]$.

In the main result of the paper (Theorem 1 below) we will use the following definition (Levant, 2005a): The origin $x = 0$ of a differential inclusion $\dot{x} \in F(x)$ (a differential equation $\dot{x} = f(x)$) is called *globally uniformly finite-time stable* if it is Lyapunov stable and, for any $R > 0$, there exists $T > 0$ such that any trajectory with initial condition $\|x_0\| < R$ reaches 0 at time T and $x(t) \equiv 0$ for all $t \geq T$.

Theorem 1. *Consider the third-order system (4) with a uniformly bounded signal $|\dot{\mu}(t)| \leq \Delta$. Then, for every $\Delta \geq 0$ and $\alpha > 0$, there exist (positive) values of the gains (k_1, k_2, L) such that the state x converges to zero globally, uniformly and in finite-time, despite any bounded perturbation $|\dot{\mu}(t)| \leq \Delta$.*