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Brief paper

Properties of feedback Nash equilibria in scalar LQ differential games ☆

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ABSTRACT

In this paper we study the scalar linear quadratic differential game with state feedback information structure. Using a geometric approach, we present a complete characterization when this game will have no, one or multiple equilibria. Furthermore, we investigate the effect on this solution structure of some characteristics of the game, i.e., the number of players; the entrance of new players; the level of asymmetry; and the impact entrance of an additional player has on the closed-loop stability of the game. For that purpose we distinguish three types of the game: the economic game; the regulator game and the mixed game. The analysis is restricted to the case the involved cost depend only on the state and control variables.

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1. Introduction

In the last decades, there is an increased interest in studying diverse problems in economics and engineering using dynamic games. In particular, in environmental economics and macroeconomic policy coordination, dynamic games are a natural framework to model policy coordination problems (see e.g. the books and references in Dockner, Jørgensen, Van Long, and Sorger (2000), Grass, Caulkins, Feichtinger, Tragler, and Behrens (2008), Jørgensen and Zaccour (2003) and Plasmans, Engwerda, van Aarle, Di Bartolomeo, and Michalak (2006)). In engineering, the theory is used to model problems in, e.g., finance, robust optimal control and pursuit-evasion problems. Particularly in the area of robust optimal control, the theory of linear quadratic differential games has been extensively developed (see, e.g., Başar & Bernhard, 1995; Engwerda, 2005; Kun, 2001; Mukaidani, 2009). In engineering, using this framework, applications are reported from diverse areas: robot control formation (Gu, 2006); interconnection of electric power systems (Mukaidani, 2009); multipath routing in communication networks (Altman & Başar, 1998; Lin, Wang, Zhou, & Miao, 2010); solving mixed H_2/H_∞ control problems (Limebeer, Anderson, & Hendel, 1994); military operations of autonomous vehicles (Li & Cruz, 2011).

In linear quadratic differential games, the environment is modeled by a set of linear differential equations and the objectives are modeled using quadratic functions. Assuming that players do not cooperate and look for linear feedback strategies which lead to a worse performance if they unilaterally deviate from it, leads to the study of so-called linear feedback Nash equilibria (FNE). The resulting equilibrium strategies have the important property that they are strong time consistent. A property which, e.g., does not hold under an open-loop information structure (see, e.g., Başar & Olsder, 1999, chap. 6.5). Under a feedback information structure also nonlinear strategies may occur. However, in many applications there is a preference for the use of linear FNE strategies. For that reason we just consider linear FNE strategies in this paper.

This problem has been considered by many authors and dates back to the seminal work of Starr and Ho (1969). For the fixed finite planning horizon there exists at most one FNE (see, e.g., Lukes, 1971). For an infinite planning horizon, the affine-quadratic differential game is solved in Engwerda and Salmah (2013). To find the FNE in this game involves solving a set of coupled algebraic Riccati-type equations (ARE). Only a few existence results are known for some special cases of these equations (see, e.g., Abou-Kandil, Freiling, Ionescu, & Jank, 2003; Engwerda, 2005; Papavassilopoulos, Medanic, & Cruz, 1979). It has been shown (see, e.g., Engwerda, 2005; Papavassilopoulos & Olsder, 1984) that the number of equilibria can vary between zero and infinity. Clearly, both from a computational point of view, and to have a better understanding of the qualitative properties of this game, one would like to characterize the number of equilibria as a function of the model parameters. Particularly in the context of large scale systems, it seems interesting to have parametric conditions

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under which this game has no, or, a unique equilibrium. Also, this may be helpful in the assessment of calculating the cost/gains of different information/cooperation structures (see, e.g., Başar & Zhu, 2011), or, in finding areas where approximate solutions of certain nonlinear differential games exist (see, e.g., Mylvaganam, Sassano, & Astolfi, 2015). For the most simple linear two-player scalar case, where the performance criterion is a strict positive quadratic function of both states and controls, such an analysis is performed in Engwerda (2000a,b) and generalized in Engwerda (2005, chap. 8.4). There it is shown that this game can have zero up to three different equilibria, and parametric conditions are provided for each case. Here we extend this approach for the N-player case. Section 2 recalls from Engwerda (2005) the basic model and approach. In Section 3 we show under which conditions the game will have either no, one, or multiple equilibria. Furthermore, as a by-product of the approach, we point out the effect entrance/drop out of players has on the closed-loop system stability if the game has a unique equilibrium. The existence conditions provided in Section 3 are primarily presented in geometric terms. Section 4 uses them to obtain analytic existence conditions in the symmetric case and elaborates this case. Section 5, next, analysis in a two-player context, how asymmetry between players affects the area where a unique equilibrium exists. Finally, Section 6 reviews a number of obtained results and indicates some directions for future research. Proofs of most results are directed to the Appendix.

2. Preliminaries

In this paper we consider the problem where N players try to minimize their performance criterion in a non-cooperative setting. Each player controls a different input in a single system. The system is described by the following scalar differential equation

$$\dot{x}(t) = ax(t) + \sum_{i=1}^{N} b_i u_i(t), \qquad x(0) = x_0. \tag{1}$$

Here x is the state of the system, u_i is the scalar variable player i can manipulate, x_0 is the arbitrarily chosen initial state of the system, a (the state feedback parameter), b_i, $i \in \mathbf{N} := \{1, \ldots, N\}$, are constant system parameters, and $\dot{x}$ denotes the time derivative of x. The aim of player $i \in \mathbf{N}$ is to minimize:

$$J_i(u_1, \ldots, u_N) := \int_0^{\infty} \{q_i x^2(t) + r_i u_i^2(t)\} dt. \tag{2}$$

Here $r_i > 0$ and both b_i and q_i differ from zero. So, player i is not directly concerned about the control efforts player j uses to manipulate the system. This assumption is crucial for the analysis below. Players act non-cooperatively and use time invariant feedback strategies, $u_i(t) = f_i x(t)$, to control the system. This, on the understanding, they do not want to destabilize the system. So, the set of strategies is restricted to

$$\mathcal{F}_N := \left\{ (f_1, \ldots, f_N) \mid a + \sum_{i=1}^{N} b_i f_i < 0 \right\}.$$

This restriction is essential. Indeed, FNE exist in which a player can improve unilaterally by choosing a feedback for which the closed-loop system is unstable (see Mageirou, 1976). Any $f \in \mathcal{F}$ is called *stabilizing*. A set of feedback strategies is called a Nash equilibrium if none of the players can improve his performance by unilaterally choosing a different strategy within the set $\mathcal{F}$. More formally, using the notation $\bar{f}_{-i}(f_i) := (\bar{f}_1, \ldots, \bar{f}_{i-1}, f_i, \bar{f}_{i+1}, \ldots, \bar{f}_N)$:

Definition 2.1. The N-tuple $\bar{f} := (\bar{f}_1, \ldots, \bar{f}_N)$ is called a set of *(linear stabilizing stationary) feedback Nash equilibrium* strategies if, for all $i \in \mathbf{N}$, the following inequalities hold:

$$J_i(\bar{f}, x_0) \le J_i(\bar{f}_{-i}(f_i), x_0),$$

for all initial states x_0, and for all $f_i \in \mathbb{R}$ such that $\bar{f}_{-i}(f_i) \in \mathcal{F}_{\mathcal{N}}$. □

Below we drop the adjectives linear, stabilizing and stationary in the above definition and use the shorthand notation FNE to denote the by these actions implied equilibrium cost, and the actions themselves as FNE actions or strategies.

We can assume here, without loss of generality, that r_i are positive and both b_i and q_i differ from zero. For, in case $r_i \le 0$, the problem has no solution; and in case either $b_i = 0$ or $q_i = 0$, the optimal control for player i is to use no control, i.e. $u_i(.) = 0$, at any point in time. So, in the last mentioned case, the player could be discarded from the game. For this game we distinguish three cases.

Definition 2.2. Game (1), (2) is called a regulator game if in (2) $q_i > 0$, $i \in \mathbf{N}$; an economic game if in (2) $q_i < 0$, $i \in \mathbf{N}$; a mixed game otherwise. □

The attached names are inspired by the fact that, in case $q_i > 0$, $i \in \mathbf{N}$, the game can be interpreted as a problem where all players like to track the system's state, x, as fast as possible to zero using as less as possible control efforts, u_i. Whereas in case $q_i < 0$, $i \in \mathbf{N}$, the game can be interpreted as a game between players who all like to maximize their profits (measured by the state variable x) using their input (measured by u_i) as efficient as possible.

The FNE for game (1), (2) are determined by the solutions of a set of coupled algebraic Riccati equations (ARE). With $s_i := \frac{b_i^2}{r_i}$ these equations in the variables k_i reduce to (see, e.g., Engwerda, 2005):

$$\left(a - \sum_{j=1}^{N} k_j s_j\right) k_i + k_i \left(a - \sum_{j=1}^{N} s_j k_j\right) + q_i + k_i s_i k_i = 0, \quad i \in \mathbf{N}. \tag{3}$$

The precise statement is as follows:

Theorem 2.3. *Game (1), (2) has a FNE if and only if (iff.) there exist N scalars k_i such that (3) holds and $a - \sum_{j=1}^{N} s_j k_j < 0$. If this condition holds, the N-tuple $(\bar{f}_1, \ldots, \bar{f}_N)$ with $\bar{f}_i := -r_i^{-1} b_i k_i$ is a FNE and $J_i(\bar{f}_1, \ldots, \bar{f}_N, x_0) = k_i x_0^2$.* □

So, to determine FNE we have to find all stabilizing solutions of (3). To find these solutions, following Engwerda (2005, Section 8.5.1), we introduce next variables:

$$\sigma_i := s_i q_i, \quad y_i := s_i k_i, \quad i \in \mathbf{N}, \quad \text{and}$$

$$y_{N+1} := -a_{cl} := -\left(a - \sum_{j=1}^{N} y_j\right).$$

Note that, by relabeling the player indices, we can enforce that $\sigma_1 \ge \cdots \ge \sigma_N$. This ordering is assumed to hold throughout. Furthermore, since $f_i = \frac{-1}{b_i} y_i$, there is a bijection between $(f_1, \ldots, f_N)$ and $(y_1, \ldots, y_N)$. Using this notation, (3) can be rewritten as

$$y_i^2 - 2y_{N+1} y_i + \sigma_i = 0, \quad i \in \mathbf{N}. \tag{4}$$

So, FNE exist iff. above N quadratic equations, and the equation

$$y_{N+1} = -a + \sum_{j=1}^{N} y_j, \tag{5}$$

have a real solution y_i, $i \in \mathbf{N}$, with $y_{N+1} > 0$. The solutions of (4) are $y_i = y_{N+1} + \sqrt{y_{N+1}^2 - \sigma_i}$ and $y_i = y_{N+1} - \sqrt{y_{N+1}^2 - \sigma_i}$, $i \in \mathbf{N}$. Substitution of this into (5) yields next result (see Appendix).

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