Automatica 69 (2016) 176–180

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Technical communique

Continuous-time mean–variance portfolio selection with random horizon in an incomplete market[☆]

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ARTICLE INFO

Article history:
Received 20 March 2015
Received in revised form
3 December 2015
Accepted 28 January 2016

Keywords:
Mean–variance model
Stochastic LQ control
Backward stochastic differential equation
BMO-martingale

ABSTRACT

In this paper, we consider a continuous-time mean–variance portfolio selection problem with random market parameters and random time horizon in an incomplete market. This problem will be formulated as a linearly constrained stochastic linear quadratic (LQ) optimal control problem. The solvability of this LQ problem will be reduced to the global solvability of two backward stochastic differential equations (BSDEs). One is conventionally called a stochastic Riccati equation (SRE), and the other is referred to as an auxiliary BSDE. We shall apply the martingales of bounded mean oscillation, briefly called BMO-martingales, to provide a direct and simplified proof of the solvability of the two BSDEs. We also derive closed-form expressions for both the optimal portfolios and the efficient frontier in terms of the solutions of the two BSDEs.

1. Introduction

Mean–variance portfolio selection problem is concerned about the tradeoff between the terminal return and the associated risk of an investment among a number of securities. It was first proposed and solved in the single-period setting by Markowitz (1952), and then extended to multi-period case by Li and Ng (2000). Another milestone is the work of Zhou and Li (2000), in which a continuous-time situation was studied. The significant contribution of Zhou and Li (2000) is that it provided an appropriate and effective framework in terms of stochastic LQ controls for the continuous-time mean–variance problems. In Zhou and Li (2000), all the market parameters are assumed to be deterministic. Motivated by the need of more realistic models, Lim and Zhou (2002) solved a mean–variance problem with random market parameters in a complete market. Along this line, Lim (2004) and Yu (2013) considered the same problem, but in an incomplete market, or with a random time horizon (see also Kharroubi, Lim, & Ngoupeyou, 2013), respectively.

An important theoretical innovation of Lim (2004) and Yu (2013) is the proofs of global solvability of the SREs and the auxiliary linear BSDEs. The SREs are important tools in LQ control theory and there exist many results on their solvability (see, for example, Kohlmann & Tang, 2002, 2003). However, there were no corresponding results which covered the situations studied by Lim (2004) and Yu (2013). With the help of BSDEs with quadratic growth and some other techniques, the authors proved the existence and uniqueness of the SREs, but their proofs are intricate. On the other hand, for the auxiliary BSDE in Lim (2004), although linear, the existence and uniqueness are not evident because the coefficients of this BSDE are only square integrable rather than uniformly bounded. In order to prove this issue, the author used the variance optimal martingale measure (VMM) making the proof abstract.

Based on Lim (2004) and Yu (2013), in this paper we consider a continuous-time mean–variance portfolio selection problem with random market parameters and a random time horizon in an incomplete market. In the present case, the corresponding SRE and auxiliary BSDE become more complicated so that both the methods of Lim (2004) and Yu (2013) cannot be used. Inspired by Hu, Imkeller, and Müller (2005) and Hu, Jin, and Zhou (2012), we overcome this difficulty. More importantly, by virtue of BMO-martingales, our proof for the solvability of the two BSDEs is direct and much simplified. Then we give analytic expressions for the efficient portfolios in feedback forms as well as the efficient frontier.

☆ This work is supported by the National Natural Science Foundation of China (11471192 and 61573217), 111 project (B12023), the Natural Science Foundation of Shandong Province (JQ201401, ZR2015JL003) and the Chang Jiang Scholar Program of Chinese Education Ministry. The material in this paper was not presented at any conference. This paper was recommended for publication in revised form by Associate Editor Michael V. Basin under the direction of Editor André L. Tits.

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http://dx.doi.org/10.1016/j.automatica.2016.02.017

2. Market model and problem formulation

Let $T > 0$ be the end of a finite time horizon and $(\Omega, \mathcal{A}, \{\mathcal{F}_t\}_{t\in[0,T]}, P)$ be a complete filtered probability space. Let $\overline{W}(t) = (W(t)', B(t)')' = (W_1(t), \ldots, W_m(t), B_1(t), \ldots, B_d(t))'$ for $m \geq 1$ and $d \geq 0$ be an $(m+d)$-dimensional standard Brownian motion defined on this space. We further assume that the filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$ with $\mathcal{F}_T \subset \mathcal{A}$ is generated by $\overline{W}(t)$, augmented by all P-null sets in $\mathcal{A}$ so that $t \mapsto \mathcal{F}_t$ is continuous. In this paper, as same as Lim (2004), we use $B(t)$ to model the market incompleteness with $d = 0$ corresponding to the case of a complete market.

Throughout this paper, for R^n-valued $\mathcal{F}_t$-adapted processes, we denote the set of square integrable processes by $L^2_{\mathcal{F}}(0, T; R^n)$, the set of continuous processes $f(t)$ such that $E\{\sup_{t\in[0,T]} |f(t)|^2\} < \infty$ by $L^2_{\mathcal{F}}(\Omega; C(0, T; R^n))$, and the set of uniformly bounded continuous processes by $L^\infty_{\mathcal{F}}(\Omega; C(0, T; R^n))$.

In a financial market consisting of a bond and m stocks traded continuously, consider an agent invests (in a self-financing way) at time t the amount $u_i(t)$ of the wealth $x(t)$ in the ith security, $i = 0, 1, \ldots, m$. Then, the wealth $x(t)$ with the initial endowment x_0 evolves according to the following SDE on $[0, T]$:

$$\begin{cases} dx(t) = \{r(t)x(t) + b(t)u(t)\}dt + u(t)'\sigma(t)dW(t), \\ x(0) = x_0 > 0, \end{cases} \tag{1}$$

where the interest rate $r(t) \geq 0$, the appreciation rate $\mu_i(t)$ and volatility rate $\sigma_i(t) = (\sigma_{i1}(t), \ldots, \sigma_{im}(t))$ of the ith stock are uniformly bounded $\mathcal{F}_t$-adapted scalar-valued stochastic processes. We assume that the volatility matrix $\sigma(t) = (\sigma_{ij}(t))_{i,j=1,2,\ldots,m}$ is uniformly non-degenerate (that is, there exists a constant $\delta > 0$ such that $\sigma(t)\sigma(t)' \geq \delta I_m$, where I_m is the $m \times m$ identity matrix), and denote $b(t) = (\mu_1(t) - r(t), \ldots, \mu_m(t) - r(t))$, $u(t) = (u_1(t), \ldots, u_m(t))'$. By convention, we call $u(t)$ a portfolio of the agent.

Definition 2.1. A portfolio $u(t)$ is said to be admissible if $u(t) \in L^2_{\mathcal{F}}(0, T; R^m)$.

In this paper, we assume that the agent's exit time τ is a positive random variable measurable with respect to $\mathcal{A}$, which may strictly bigger than $\mathcal{F}_T$. Then the random exit time relies not only on the asset prices, but also on other factors. As in Blanchet-Scalliet, El Karoui, Jeanblanc, and Martellini (2008) and Yu (2013), we will use the method of separation to handle the random exit time as follows: Conditioning up $\mathcal{F}_t$ contains the information of the asset prices up to time t. We denote by $F(t) = P(\tau \leq t|\mathcal{F}_t)$, the conditional distribution function of timing uncertainty. It is easy to verify that $F(t)$ is an $\mathcal{F}_t$-submartingale and the function $t \mapsto E[F(t)]$ is right-continuous, then $F(t)$ has a right-continuous modification. From the Doob–Meyer decomposition, we have $F(t) = M(t) + A(t)$, where $M(t)$ is a martingale and $A(t)$ is an increasing process. We further make the following assumptions.

Assumption 2.2. The process $A(t)$ is absolutely continuous with respect to Lebesgue's measure, with a bounded non-negative density denoted by $a(t)$, i.e., $A(t) = \int_0^t a(s)ds$.

Under Assumption 2.2, we obtain immediately the boundedness of $A(t)$, and then of the martingale $M(t)$. From the martingale representation theorem, there exists a unique process $m(t) \in L^2_{\mathcal{F}}(0, T; R^{m+d})$ such that $M(t) = \int_0^t m(s)'d\overline{W}(s)$.

Assumption 2.3. There exists a positive constant C such that $\int_0^T |m(t)|^2 dt \leq C$.

Assumption 2.4. There exists a positive constant ε such that $F(T) \leq 1 - \varepsilon$.

Suppose the deadline of the investment is T after which the agent can no longer trade the assets any way. Thus, the actual exit time of the agent is $T \wedge \tau$ and his objective is to find an admissible portfolio $u(t)$, among all such admissible portfolios whose expected terminal wealth $E[x(T \wedge \tau)] = z$, for some given $z \in R$, so that the risk measured by the variance of the terminal wealth $\mathrm{Var}[x(T\wedge\tau)] = E[x(T\wedge\tau) - E[x(T\wedge\tau)]]^2 = E[x(T\wedge\tau) - z]^2$ is minimized.

From the definition of $F(t) = P(\tau \leq t|\mathcal{F}_t)$, its decomposition $F(t) = M(t) + A(t)$ and Assumptions 2.2, 2.3, we have $E[x(T \wedge \tau)] = E[\int_0^T a(t)x(t)dt + (1 - F(T))x(T)]$ and $\mathrm{Var}[x(T \wedge \tau)] = E[\int_0^T a(t)(x(t) - z)^2 dt + (1 - F(T))(x(T) - z)^2]$.

Definition 2.5. Under Assumptions 2.2, 2.3, 2.4, the mean–variance portfolio selection problem with random market parameters and a random horizon in an incomplete market is formulated as a constrained stochastic optimal control problem, parameterized by $z \in R$:

$$\begin{cases} \text{minimize } J_{MV}(u(\cdot)) = E\left[\int_0^T a(t)(x(t) - z)^2 dt \right. \\ \qquad\qquad \left. + (1 - F(T))(x(T) - z)^2\right], \\ \text{subject to } \begin{cases} J_1(u(\cdot)) = E\left[\int_0^T a(t)x(t)dt \right. \\ \qquad\qquad \left. + (1 - F(T))x(T)\right] = z, \\ (x(\cdot), u(\cdot)) \quad \text{admissible}. \end{cases} \end{cases} \tag{2}$$

Moreover, an admissible portfolio $u(\cdot) \in \mathcal{U}$ is said to be a feasible portfolio if it satisfies the constraint $J_1(u(\cdot)) = z$. If there exists a feasible portfolio, then problem (2) is said to be feasible. If the infimum of $J_{MV}(u(\cdot))$ is achieved by a feasible portfolio $u^*(\cdot)$, then problem (2) is said to be solvable and $u^*(\cdot)$ is called an optimal or efficient portfolio corresponding to z. The pair $(\mathrm{Var}[x(T \wedge \tau)], z) \in R^2$ is called an efficient point and the set of all the efficient points is called the efficient frontier.

With a minor modification of the proof of Proposition 3.4 in Lim and Zhou (2002), we can prove the following feasibility result.

Proposition 2.6. *Let $(\psi, \xi_1, \xi_2) \in L^2_{\mathcal{F}}(\Omega; C(0, T; R)) \times L^2_{\mathcal{F}}(0, T; R^m) \times L^2_{\mathcal{F}}(0, T; R^d)$ denotes the unique solution of the following BSDE:*

$$\begin{cases} d\psi(t) = -(r(t)\psi(t) + a(t))dt + \xi_1(t)'dW(t) \\ \qquad\quad + \xi_2(t)'dB(t), \quad t \in [0, T], \\ \psi(T) = 1 - F(T). \end{cases}$$

Then problem (2) is feasible for any $z \in R$ if and only if

$$E\left[\int_0^T \left|\psi(t)b(t) + \xi_1(t)'\sigma(t)'\right|^2 dt\right] > 0. \tag{3}$$

In the rest of this paper, we shall assume that condition (3) holds, i.e., the mean–variance problem (2) is feasible for any given $z \in R$. Under this condition, we proceed to study the issue of optimality. Note that the mean–variance problem (2) is a dynamic optimization problem with a constraint $J_1(u(\cdot)) = z$. Here we apply the Lagrange multiplier technique to handle this constraint. For each $\lambda \in R$, define:

$$\begin{aligned} J(u(\cdot), \lambda) &= J_{MV}(u(\cdot)) + 2\lambda(J_1(u(\cdot)) - z) \\ &= E\left[\int_0^T a(t)(x(t) + (\lambda - z))^2 dt \right. \\ &\qquad \left. + (1 - F(T))(x(T) + (\lambda - z))^2\right] - \lambda^2. \end{aligned} \tag{4}$$