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Algebraic criteria of global observability of polynomial systems☆

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ABSTRACT

Global observability of continuous-time polynomial systems is studied. An algebraic necessary and sufficient condition of global observability is proved. It is expressed with the aid of real radicals of ideals in the ring of polynomials and is based on the real theorem of zeros from real algebraic geometry.

1. Introduction

Global observability is one of the fundamental properties of control systems. For analytic systems it is equivalent to injectivity of a family of analytic functions defined on the state space of the system (see Hermann & Krener, 1977). If the system is polynomial, this family may be reduced to a finite one and the tools from algebraic geometry may be used. In Kawano and Ohtsuka (2013) and Tibken (2004) these tools were used to derive geometric criteria of global observability of polynomial systems. In Kawano and Ohtsuka (2013) also some sufficient algebraic conditions were proposed. We follow these ideas in this paper making one step further: we present necessary and sufficient algebraic conditions of global observability and global observability at a particular point for polynomial systems.

We draw on ideas and results of real algebraic geometry (see e.g. Bochnak, Coste, & Roy, 1998). In particular the real theorem of zeros (real *Nullstellensatz*) – a fundamental result of real algebraic geometry – is used to prove the main results of this paper. The main concept that appears in the study is the real radical of an ideal of the ring of polynomials. It is different from ordinary radical, which appears in Hilbert's *Nullstellensatz* in complex algebraic geometry (over the field of complex numbers). The ordinary radical was used in Kawano and Ohtsuka (2013) and Tibken (2004), but it is not a proper tool in the analysis of the systems over real numbers. It is harder to compute the real radical than the ordinary radical, but we provide several examples showing how to check the conditions of observability in which real radicals appear. We also comment on algorithms of computing real radicals.

Real radicals were already successfully used in characterizing local and stable (robust) local observability of analytic continuous-time and discrete-time systems (see Bartosiewicz, 1995, 1998, 1999). These results have been recently extended to systems on time scales in Bartosiewicz (2013), which encompass both continuous-time and discrete-time systems. The present paper may be seen as a passage from local to global problems, where the same tools are used. The main difference concerns the rings that appear in these two areas. In local observability the ring of germs of analytic functions was used, while for global observability we exploit the ring of global polynomial functions.

2. Global observability

Let us consider a control system with output, denoted by Σ:

$$\begin{aligned} x'(t) &= f(x(t), u(t)) \\ y(t) &= h(x(t)), \end{aligned} \tag{1}$$

where $t \in \mathbb{R}$, $x(t) \in \mathbb{R}^n$, $u(t) \in \Omega$ — an arbitrary set (the set of control values), $y(t) \in \mathbb{R}^p$. We assume that for every $\omega \in \Omega$, f_ω defined by $f_\omega(x) := f(x, \omega)$ is a polynomial vector field on $\mathbb{R}^n$. It may be identified with a map from $\mathbb{R}^n$ to $\mathbb{R}^n$ with polynomial components. Similarly h is assumed to be a polynomial map. Moreover, we assume that controls are piecewise constant functions from $[0, +\infty)$ into Ω.

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Let $x(t, x_0, u)$ denote the solution of (1) satisfying $x(0) = x_0$, corresponding to control u and evaluated at $t \geq 0$. Let $x_1, x_2 \in \mathbb{R}^n$. Then x_1 and x_2 are called *indistinguishable* if

$$h(x(t, x_1, u)) = h(x(t, x_2, u))$$

for every control u and for every $t \geq 0$ for which both sides of the equation are defined. Otherwise x_1 and x_2 are *distinguishable*.

Let us recall the definitions of the basic concepts.

System Σ is *globally observable at point* $x_0 \in \mathbb{R}^n$ iff for any $x \in \mathbb{R}^n$ different from x_0, x_0 and x are distinguishable.

System Σ is *globally observable* iff it is globally observable at every point $x_0 \in \mathbb{R}^n$. In other words, system Σ is *globally observable* iff every two distinct states in $\mathbb{R}^n$ are distinguishable.

Remark 1. There are many different concepts of observability for nonlinear systems. For example, *complete uniform observability*, a stronger concept than global observability, is used in Gauthier, Hammouri, and Othman (1992) to construct an observer. Global observability studied here is a classic system-theoretic concept, used for example in studies on minimal realizations. It implies possibility of reconstruction of the state from the output. Since no rank condition is involved, the function that assigns the initial state to the output trajectory may have lower regularity than the system.

Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be polynomial and g be a polynomial vector field on $\mathbb{R}^n$. Then $L_g\varphi := \nabla\varphi \cdot g$ denotes the Lie derivative of φ with respect to g. Observe that $L_g\varphi$ is again a polynomial function.

Let $H(\Sigma)$ denote the set of all functions of the form $L_{f_{\omega_k}} \dots L_{f_{\omega_1}} h_i$ for $i = 1, \dots, p$, $k \geq 0$ and $\omega_j \in \Omega$ for $j = 1, \dots, k$, where for $k = 0$ such a function is just h_i.

Theorem 2 (*Hermann & Krener, 1977*)**.** *The points x_1 and x_2 are indistinguishable if and only if $\varphi(x_1) = \varphi(x_2)$ for all $\varphi \in H(\Sigma)$.*

Corollary 3. *System Σ is globally observable at x_0 iff for every $x \in \mathbb{R}^n$ different from x_0 there is $\varphi \in H(\Sigma)$ s.t. $\varphi(x_0) \neq \varphi(x)$.*

System Σ is globally observable iff for every distinct $x_1, x_2 \in \mathbb{R}^n$ there is $\varphi \in H(\Sigma)$ s.t. $\varphi(x_1) \neq \varphi(x_2)$.

3. Geometric characterization

We recall here geometric characterizations of global observability. These characterizations will allow us to pass to algebraic criteria. They were proved earlier in Kawano and Ohtsuka (2013).

For a set G of real polynomial functions on $\mathbb{R}^n$ let us define the *zero-set* of G by

$$\mathcal{Z}(G) := \{x \in \mathbb{R}^n : \varphi(x) = 0 \text{ for every } \varphi \in G\}.$$

Let us first establish a geometric characterization of global observability at a point.

Let us fix $x_0 \in \mathbb{R}^n$ and denote by $H(\Sigma)_{x_0}$ a family of polynomial functions on $\mathbb{R}^n$ of the form $\varphi - \varphi(x_0)$, where $\varphi \in H(\Sigma)$. Note that all functions from $H(\Sigma)_{x_0}$ vanish at x_0.

Proposition 4. *Σ is globally observable at x_0 if and only if $\mathcal{Z}(H(\Sigma)_{x_0}) = \{x_0\}$.*

Let $I(\Sigma)_{x_0}$ denote the ideal of the ring of polynomials $\mathbb{R}[x_1, \dots, x_n] =: \mathbb{R}[x]$ generated by $H(\Sigma)_{x_0}$. It consists of polynomials of the form $\alpha_1\varphi_1 + \dots + \alpha_k\varphi_k$, where $k \in \mathbb{N}$, $\alpha_1, \dots, \alpha_k \in \mathbb{R}[x]$ and $\varphi_1, \dots, \varphi_k \in H(\Sigma)_{x_0}$. Observe that all polynomials from this ideal vanish at x_0 and $H(\Sigma)_{x_0} \subset I(\Sigma)_{x_0}$.

Corollary 5. *Σ is globally observable at x_0 if and only if $\mathcal{Z}(I(\Sigma)_{x_0}) = \{x_0\}$.*

Remark 6. In general the set of functions $H(\Sigma)_{x_0}$ is infinite, but the ideal $I(\Sigma)_{x_0}$ generated by $H(\Sigma)_{x_0}$ is finitely generated. This follows from the Hilbert basis theorem (Atiyah & Macdonald, 1969). Moreover, one can show that the generators of the ideal could be taken from the set $H(\Sigma)_{x_0}$. However the number of these generators cannot be easily deduced from the system.

The *observation ideal* of system Σ is the ideal $I(\Sigma)$ of the ring $\mathbb{R}[x, z] := \mathbb{R}[x_1, \dots, x_n, z_1, \dots, z_n]$ generated by functions $\varphi(x) - \varphi(z)$, where $\varphi \in H(\Sigma)$.

Let D_n denote the diagonal of $\mathbb{R}^{2n}$, i.e. $D_n = \{(x, x) \in \mathbb{R}^n \times \mathbb{R}^n : x \in \mathbb{R}^n\}$.

The following fact follows directly from the definition:

Proposition 7. $\mathcal{Z}(I(\Sigma)) \supseteq D_n$.

Let us state now a geometric characterization of global observability.

Proposition 8. *Σ is globally observable iff $\mathcal{Z}(I(\Sigma)) = D_n$.*

4. Algebraic criteria

Let I be an ideal of a commutative ring R. The *radical* of I, denoted by $\sqrt{I}$, is the set of all $a \in R$ such that for some $k \in \mathbb{N}$, $a^k \in I$. The *real radical* of I, denoted by $\sqrt[\mathbb{R}]{I}$, is the set of all $a \in R$ for which there are $m \in \mathbb{N}$, $k \in \mathbb{N}$ and $b_1, \dots, b_k \in R$ such that

$$a^{2m} + b_1^2 + \dots + b_k^2 \in I.$$

Proposition 9. *The real radical has the following properties:*

1. *The real radical of I is an ideal of R.*
2. $I \subseteq \sqrt{I} \subseteq \sqrt[\mathbb{R}]{I}$.
3. *If I is proper, then $\sqrt[\mathbb{R}]{I}$ is proper.*
4. *If $I \subseteq J$, then $\sqrt[\mathbb{R}]{I} \subseteq \sqrt[\mathbb{R}]{J}$.*

Proof. See e.g. Bochnak et al. (1998).

Example 10. Let $R = \mathbb{R}[x_1, x_2]$ and $I = ((x_1^2 + x_2^2)^2)$ — the ideal generated by $(x_1^2 + x_2^2)^2$. It is easy to see that $\sqrt{I} = (x_1^2 + x_2^2)$. Since $x_1^4 + 2(x_1x_2)^2 + x_2^4 \in I$, then $x_1, x_2 \in \sqrt[\mathbb{R}]{I}$. Thus $\sqrt[\mathbb{R}]{I} = (x_1, x_2)$, because (x_1, x_2) is a maximal ideal.

For a subset A of $\mathbb{R}^n$ let $\mathcal{I}(A)$ denote the *zero-ideal* of A. It is the set of all $\varphi \in \mathbb{R}[x]$ such that $\varphi(x) = 0$ for all $x \in A$. It is easy to see that $\mathcal{I}(A)$ is an ideal of the ring $\mathbb{R}[x]$.

Proposition 11. *Let I be an ideal of $\mathbb{R}[x]$ and A be a subset of $\mathbb{R}^n$. Then*

1. $\mathcal{Z}(\mathcal{I}(A)) \supseteq A$,
2. $\mathcal{Z}(\mathcal{I}(\mathcal{Z}(I))) = \mathcal{Z}(I)$,
3. $\mathcal{I}(\mathcal{Z}(I)) \supseteq I$,
4. $\mathcal{I}(\mathcal{Z}(\mathcal{I}(A))) = \mathcal{I}(A)$.

Proof. See e.g. Bochnak et al. (1998).

The following theorem is one of the fundamental facts of real algebraic geometry (see e.g. Bochnak et al., 1998). It will help us to transfer the geometric characterization of observability to an algebraic one.

Theorem 12 (*Real Nullstellensatz*)**.** *Let I be an ideal of $\mathbb{R}[x_1, \dots, x_n]$. Then $\mathcal{I}(\mathcal{Z}(I)) = \sqrt[\mathbb{R}]{I}$.*

Example 13. Let $R = \mathbb{R}[x_1, x_2]$ and $I = ((x_1^2 + x_2^2)^2)$. Then $\mathcal{Z}(I) = \{(0, 0)\}$ and $\mathcal{I}(\mathcal{Z}(I)) = (x_1, x_2)$. It coincides with $\sqrt[\mathbb{R}]{I}$ computed in Example 10.