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Stability analysis of feedback systems with dead-zone nonlinearities by circle and Popov criteria☆

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ABSTRACT

A method of global stability analysis is proposed for a feedback system with dead-zone nonlinearities. Using a global property that the output of a saturation function is bounded, the bound on the input to the saturation function is estimated using the L_∞ norm of a linear subsystem. The feedback system can be treated as a feedback system with a narrower sector bound using this method, and a sharper global stability condition is obtained by applying the circle or Popov criterion to the system.

1. Introduction

Linear systems with saturation nonlinearities occur very often in practice. These systems can be represented with dead-zone nonlinearities equivalently. Stability analysis of these systems is important. The circle and Popov criteria guarantee absolute stability for feedback systems with sector-bounded nonlinearities (Khalil, 1996), and they are widely used for stability analysis and control design of such feedback systems. Application to the anti-windup compensator design is a representative example (Zaccarian & Teel, 2011).

If the sector-bounded function contains the dead-zone function globally, global asymptotic stability of the equilibrium can be tested using the circle criterion. In the case that the circle criterion does not hold for the system, methods of estimating the domain of attraction have been studied. In Hindi and Boyd (1998) and Pittet, Tarbouriech, and Burgat (1997), the domain of attraction is estimated using quadratic and Lur'e-type Lyapunov functions based on the circle and Popov criteria, where a narrower sector-bounded nonlinearity that contains the dead-zone function locally around the origin is used. The conditions have been represented using matrix inequalities and the maximization of the size of the domain of attraction has been considered. By fixing some of the parameters, these matrix inequalities simplify to linear matrix inequalities (LMIs) (Boyd, Ghaoui, Feron, & Balakrishnam, 1994). A relation between the achievable domains of attraction derived from a linear analysis and the circle criterion has been clarified for conservativeness (Kiyama & Iwasaki, 2000).

Less conservative estimates of the domain of attraction were obtained using a quadratic Lyapunov function in Gomes da Silva and Tarbouriech (2005) and Hu, Lin, and Chen (2002). An invariance condition set by exploring the special property of the saturation nonlinearity has been developed (Hu et al., 2002), and a modified sector condition has been used (Gomes da Silva & Tarbouriech, 2005). Because these stability conditions are LMIs with respect to not only a Lyapunov variable but also a control gain matrix, they are useful in control design.

The above methods are based on Lyapunov stability and they use the property of the dead zone or saturation functions in a bounded interval around the origin. In other words, none of the methods consider any property outside the interval, without which global stability cannot be shown. Even if the circle criterion is not satisfied globally, there is a possibility that the feedback system is globally asymptotically stable. To derive a sharper global stability condition for the feedback system with dead-zone nonlinearities, in this paper, we propose a method of estimating the bound on the input of the dead-zone function using the property that the output of the saturation function is bounded globally. A sharper global

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stability condition can then be obtained by applying the circle or Popov criterion to the feedback system with the narrower sector-bounded nonlinearities.

Note that a system with dead-zone functions can be described as a piecewise affine system exactly. Sufficient stability conditions for global stability with a piecewise quadratic Lyapunov function have been given for piecewise affine systems (Johansson, 2003), and this method has been applied to a feedback system with a control input saturation. In this paper, we compare our method with the circle and Popov criteria and this piecewise method by taking numerical examples.

2. Problem setting

Consider the feedback system described by

$$\dot{x} = Ax + Bu, \tag{1}$$

$$y = Cx, \tag{2}$$

$$u = -\psi(y), \tag{3}$$

where $u \in R^m, y \in R^m, x \in R^n$. The ith row vector of the matrix C is denoted c_i, and the jth column vector of B is denoted b_j. (A, B) and (C, A) are assumed to be controllable and observable, respectively.

The plant transfer function is given by

$$P(s) = C(sI - A)^{-1}B. \tag{4}$$

$\psi(y)$ is given by

$$\psi(y) = (\psi_1(y_1), \ldots, \psi_m(y_m))^T, \tag{5}$$

where the elements are dead-zone functions described by

$$\psi_i(\sigma) = \begin{cases} 0 & |\sigma| \le 1 \\ \sigma - 1 & \sigma > 1 \\ \sigma + 1 & \sigma < -1 \end{cases} \tag{6}$$

for $\sigma \in R$.

The origin of the feedback system is an equilibrium point, and in the vicinity of the origin, the system is described by $\dot{x} = Ax$. A is assumed to be a stable matrix, which is a necessary condition for stability. We examine a condition for global stability of the feedback system. The equilibrium point $x = 0$ is a globally stable equilibrium point of $\dot{x} = f(x)$, $x(t_0) = x_0$ if it is stable and $\lim_{t\to\infty} x(t) = 0$ for all $x_0 \in R^n$ (Sastry, 1999).

The circle and Popov criteria are briefly summarized in the following. A nonlinear function $\psi_i(\sigma)$ belongs to the sector $[0, K_i]$ if, for $\sigma \in R$,

$$0 \le \frac{\psi_i(\sigma)}{\sigma} \le K_i. \tag{7}$$

The next lemma is cited from Boyd et al. (1994), where the sign of the matrix B is changed because our system is given in the form of negative feedback.

Lemma 1 (*Popov Criterion*)**.** *Suppose that $\psi_i(y_i)$ is a time-invariant memoryless nonlinearity that belongs to the sector $[0, K_i]$ for $i = 1, \ldots, m$. The origin is then globally asymptotically stable if there exist matrices X, Λ, and W that satisfy the LMIs*

$$\begin{bmatrix} \Theta & -XB + A^TC^T\Lambda + C^TKW \\ \star & -\Lambda CB - B^TC^T\Lambda - 2W \end{bmatrix} < 0, \tag{8}$$

$$X = X^T > 0, \tag{9}$$

$$W = \mathrm{diag}(w_1, \ldots, w_m) > 0, \tag{10}$$

$$\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_m) \ge 0, \tag{11}$$

where $K = \mathrm{diag}(K_1, \ldots, K_m)$, $\Theta = XA + A^TX$, and the symbol $\star$ represents the appropriate entry resulting in an overall symmetric matrix.

According to the relation between the circle criterion and the Popov criterion (Khalil, 1996) and the matrix inequality condition of Lemma 1, the next lemma is obtained by setting $\Lambda = 0$.

Lemma 2 (*Circle Criterion*)**.** *Suppose that $\psi_i(y_i, t)$ is a memoryless, possibly time-varying, nonlinearity, which belongs to the sector $[0, K_i]$ for $i = 1, \ldots, m$. The origin is then globally asymptotically stable if the condition of Lemma 1 is satisfied with $\Lambda = 0$.*

Because the dead-zone function (6) belongs to a sector $[0, 1]$ globally, global stability conditions can be derived using the circle and Popov criteria. These criteria give sufficient conditions for global stability for all the nonlinear functions that satisfy the sector condition. Therefore, the stability conditions tend to be conservative for specific nonlinearities such as those of saturation or the dead zone. Our aim is to give a sharper condition for the global stability of feedback systems with dead-zone nonlinearities using a global property of saturation functions.

3. Main result

First, we represent the dead-zone function $\psi_i(\sigma)$ with a saturation function $\phi_i(\sigma)$. Namely,

$$\psi_i(\sigma) = \sigma - \phi_i(\sigma), \tag{12}$$

where

$$\phi_i(\sigma) = \begin{cases} \sigma & |\sigma| \le 1 \\ 1 & \sigma > 1 \\ -1 & \sigma < -1. \end{cases} \tag{13}$$

The feedback system is then expressed as

$$\dot{x} = A_cx + Bd, \tag{14}$$

$$y = Cx, \tag{15}$$

$$d = \phi(y), \tag{16}$$

where

$$A_c = A - BC \tag{17}$$

$$\phi(y) = (\phi_1(y_1), \ldots, \phi_m(y_m))^T. \tag{18}$$

We assume that A_c is a stable matrix. This assumption is reasonable because the feedback system is not globally stable for unstable A_c. The bound on the output y in the steady state is estimated according to the next lemma.

Lemma 3. *Let A_c be a stable matrix. For any initial value $x(0)$ and an arbitrarily small positive number δ, there exists a finite time $T > 0$ such that, for $t \ge T$,*

$$|y_i(t)| \le \delta + \sum_{j=1}^{m} \int_0^t |c_ie^{A_c(t-\tau)}b_j| d\tau \quad i = 1, 2, \ldots, m. \tag{19}$$

Proof. From (14) and (15),

$$y_i(t) = c_ie^{A_ct}x(0) + \sum_{j=1}^{m} \int_0^t c_ie^{A_c(t-\tau)}b_jd_j(\tau)d\tau \tag{20}$$

for $i = 1, 2, \ldots, m$. Because A_c is a stable matrix, there exists a finite $T > 0$ for any $\delta > 0$ for which

$$|c_ie^{A_ct}x(0)| \le \delta, \quad t \ge T \tag{21}$$

is satisfied. Therefore, for $t \ge T$,

$$|y_i(t)| \le \delta + \sum_{j=1}^{m} \int_0^t \left|c_ie^{A_c(t-\tau)}b_jd_j(\tau)\right| d\tau. \tag{22}$$