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Distributed robust adaptive equilibrium computation for generalized convex games ☆

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ABSTRACT

This paper considers a class of generalized convex games where each player is associated with a convex objective function, a convex inequality constraint and a convex constraint set. The players aim to compute a Nash equilibrium through communicating with neighboring players. The particular challenge we consider is that the component functions are unknown *a priori* to associated players. We study two distributed computation algorithms and analyze their convergence properties in the presence of data transmission delays and dynamic changes of network topologies. The algorithm performance is verified through demand response on the IEEE 30-bus Test System. Our technical tools integrate convex analysis, variational inequalities and simultaneous perturbation stochastic approximation.

1. Introduction

Recent advances on information technologies facilitate real-time message exchanges and decision-making among geographically dispersed strategic entities. This has boosted the emergence of new generation of networked systems; e.g., the smart grid and intelligent transportation systems. These networked systems share some common features: on one hand, the entities do not belong to a single authority and may pursue different or even competitive interests; on the other hand, each entity keeps private information which is unaccessible to others. It is of great interest to design practical mechanisms which allow for efficient coordination of self-interested entities and ensure network-wide performance. Game theory along with its distributed computation algorithms represents a promising tool to achieve the goal.

In many applications, distributed computation is executed in uncertain environments. For example, mobile robots are deployed in an operating environment where environmental distribution functions are unknown to robots in advance; e.g., Stankovic, Johansson, and Stipanovic (2012) and Zhu and Martínez (2013b). In traffic pricing, pricing policies of system operators may not be available to drivers. In optimal power flow control, the structural parameters of power systems are of national security interest and kept confidential from the public. The absence of such information makes game components; e.g., objective and constraint functions, inaccessible to players. Very recently, the informational constraint has been stimulating the investigation of *adaptive* algorithms, including Frihauf, Krstic, and Başar (2011), Liu and Krstic (2011), Stankovic et al. (2012) for continuous games and Marden, Young, Arslan, and Shamma (2009), Zhu and Martínez (2013b) for discrete games.

Literature review. Non-cooperative game theory has been widely used as a mathematical framework to reason about multiple selfish decision makers; see for instance Başar and Olsder (1982). These games have found a variety of applications in economics, communication and robotics; see Altman and Başar (1998), Chung, Hollinger, and Isler (2011), Dockner, Jorgensen, Long, and Sorger (2006), Frihauf et al. (2011) and Mitchell, Bayen, and Tomlin (2005). In non-cooperative games, decision making of individuals is inherently distributed. Very recently, this attractive feature has been utilized to synthesize cooperative control schemes, and a partial reference list for this regard includes (Arsie, Savla, & Frazzoli, 2009; Arslan, Marden, & Shamma, 2007; Li & Marden, 2011; Stankovic et al., 2012; Zhu & Martínez, 2013b).

The set of papers more relevant to our work is concerned with generalized Nash games where strategy spaces are continuous and the actions of players are coupled through utility and constraint functions. Generalized Nash games are first formulated in

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Arrow and Debreu (1954). Since then, a great effort has been dedicated to studying the existence and structural properties of generalized Nash equilibria in; e.g., Rosen (1965) and the recent survey paper (Facchinei & Kanzow, 2007). A number of algorithms have been proposed to compute generalized Nash equilibria, including ODE-based methods (Li & Basar, 1987; Rosen, 1965), nonlinear Gauss–Seidel-type approaches (Pang, Scutari, Facchinei, & Wang, 2008), iterative primal–dual Tikhonov schemes (Yin, Shanbhag, & Mehta, 2011) and best-response dynamics (Palomar & Eldar, 2010).

As mentioned, the set of papers (Frihauf et al., 2011; Liu & Krstic, 2011; Marden et al., 2009; Stankovic et al., 2012; Zhu & Martínez, 2013b) investigates the *adaptiveness* of game theoretic learning algorithms. However, none of the papers mentioned in the last two paragraphs studies the *robustness* of the algorithms with respect to network unreliability; e.g., data transmission delays, quantization and dynamically changing topologies. In contrast, the robustness has been extensively studied for consensus and distributed optimization, including, to name a few, Jadbabaie, Lin, and Morse (2003) and Nedic, Ozdaglar, and Parrilo (2010) for time-varying topologies, Rabbat and Nowak (2005) for quantization and Münz, Papachristodoulou, and Allgower (2010) for time delays. Yet the adaptiveness issue has not been addressed in this group of papers.

Contributions. In this paper, we aim to solve a class of generalized convex games over unreliable networks where the structures of component functions are unknown to the associated players. That is, we aim to simultaneously address the issues of adaptiveness and robustness for generalized convex games.

In the games, each player is associated with a convex objective function and subject to a private convex inequality constraint and a private convex constraint set. The component functions are assumed to be smooth and are unknown to the associated players. We investigate distributed first-order gradient-based computation algorithms for the following two scenarios:

[Scenario One] The game map is pseudo-monotone and the maximum delay (equivalently, the maximum number of packet dropouts or link breaks) is bounded but unknown;

[Scenario Two] The inequality constraints are absent, the (reduced) game map is strongly monotone and the maximum delay is known.

Inspired by simultaneous perturbation stochastic approximation for optimization in Spall (2003), we utilize finite differences with diminishing approximation errors to estimate first-order gradients. We propose two distributed algorithms for the two scenarios and formally prove their asymptotic convergence. The comparison of the two proposed algorithms is given in Section 5.1. The analysis integrates the tools from convex analysis, variational inequalities and simultaneous perturbation stochastic approximation. The algorithm performance is verified through demand response on the IEEE 30-bus Test System. A preliminary version of the current paper was published in Zhu and Frazzoli (2012) where the adaptiveness issue was not investigated. Due to space limitation, some proofs are omitted and can be found at Zhu and Frazzoli (2015).

2. Problem formulation

In this section, we present the generalized convex game considered in the paper. It is followed by the notions and notations used throughout the paper.

2.1. Generalized convex game

Consider the set of players $V \triangleq \{1, \ldots, N\}$ where the state of player i is denoted as $x^{[i]} \in X_i \subseteq \mathbb{R}^{n_i}$. The players are selfish and pursue different interests. In particular, given the joint state $x^{[-i]} \in X_{-i} \triangleq \prod_{j\neq i} X_j$ of its rivals,[1] each player i aims to solve the following program parameterized by $x^{[-i]} \in X_{-i}$:

$$\min_{x^{[i]}\in X_i} f_i(x^{[i]}, x^{[-i]}), \quad \text{s.t. } G^{[i]}(x^{[i]}, x^{[-i]}) \leq 0, \tag{1}$$

where $f_i : \mathbb{R}^n \to \mathbb{R}$ and $G^{[i]} : \mathbb{R}^n \to \mathbb{R}^{m_i}$ with $n \triangleq \sum_{i\in V} n_i$. In the remainder of the paper, we assume that the following properties about problem (1) hold:

Assumption 2.1. The maps f_i and $G^{[i]}$ are smooth, and the maps $f_i(\cdot, x^{[-i]})$ and $G^{[i]}(\cdot, x^{[-i]})$ are convex in $x^{[i]}$. The set X_i is convex and compact, and $X \cap Y \neq \emptyset$ where $X \triangleq \prod_{i\in V} X_i$ and $Y \triangleq \prod_{i\in V} Y_i$ with $Y_i \triangleq \{x \in X \mid G^{[i]}(x) \leq 0\}$.

We now proceed to provide an equivalent form of problem (1). To achieve this, we define the set-valued map $X_i^f : X_{-i} \to 2^{X_i}$ as follows:

$$X_i^f(x^{[-i]}) = \{x^{[i]} \in X_i \mid G^{[i]}(x^{[i]}, x^{[-i]}) \leq 0\}.$$

The set $X_i^f(x^{[-i]})$ represents the collection of feasible actions for player i when its opponents choose the joint state of $x^{[-i]} \in X_{-i}$. With the map X_i^f, problem (1) of player i is equivalent to the following one:

$$\min_{x^{[i]}\in X_i^f(x^{[-i]})} f_i(x^{[i]}, x^{[-i]}). \tag{2}$$

Given $x^{[-i]} \in X_{-i}$, each player i aims to solve problem (2). The collection of such coupled optimization problems consists of the *generalized convex game* (for short, CVX). For the CVX game, we adopt the *generalized Nash equilibrium* (for short, GNE) as the solution notion which none of the players is willing to unilaterally deviate from:

Definition 2.1. The joint state $\tilde{x} \in X \cap Y$ is a generalized Nash equilibrium of the CVX game if the following holds:

$$f_i(\tilde{x}) \leq f_i(x^{[i]}, \tilde{x}^{[-i]}), \quad \forall x^{[i]} \in X_i^f(\tilde{x}^{[-i]}),\ \forall i \in V.$$

Denote by $\mathbb{X}_{\text{CVX}}$ the set of GNEs of the CVX game. The following lemma verifies the non-emptiness of $\mathbb{X}_{\text{CVX}}$.

Lemma 2.1. *The set of generalized Nash equilibria of the CVX game is not empty, i.e.,* $\mathbb{X}_{\text{CVX}} \neq \emptyset$.

Proof. Recall that f_i is convex and $X \cap Y$ is compact. Hence, $\mathbb{X}_{\text{CVX}} \neq \emptyset$ is a direct result of Facchinei and Kanzow (2007) and Rosen (1965). •

In the CVX game, the players desire to seek a GNE. It is noted that f_i, $G^{[i]}$, and X_i are private information of player i and unaccessible to others. In order to compute a GNE, it becomes necessary that the players are inter-connected and able to communicate with each other to exchange their partial estimates of GNEs. The interconnection between players will be represented by a directed graph $\mathcal{G} = (V, \mathcal{E})$ where $\mathcal{E} \subset V \times V \setminus \text{diag}(V)$ is the set of edges. The neighbor relation is determined by the dependency of f_i and/or $G^{[i]}$ on $x^{[j]}$. In particular, $(i, j) \in \mathcal{E}$ if and only if f_i and/or $G^{[i]}$ depend upon $x^{[j]}$. Denote by $\mathcal{N}_i^{\text{IN}} \triangleq \{j \in V \mid (i, j) \in \mathcal{E}\}$ the set of in-neighbors of player i.

In this paper, we aim to develop distributed algorithms which allow for the computation of GNEs in the presence of the following two challenges.

[1] We use the shorthand $-i \triangleq V \setminus \{i\}$ throughout the paper.